

# SUSY Transformations for Quasinormal and Total-Transmission Modes of Open Systems

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Quasinormal modes are the counterparts in open systems of normal modes in conservative systems; defined by outgoing-wave boundary conditions, they have complex eigenvalues  $\omega$ . The conditions are studied for a system to have a supersymmetric (SUSY) partner with the same complex quasinormal-mode spectrum (or the same except for one eigenvalue). The discussion naturally includes total-transmission modes as well (incoming at one extreme and outgoing at the other). Several types of SUSY transformations emerge, and each is illustrated with examples, including the transformation among different Pöschl–Teller potentials and the well-known identity in spectrum between the two parity sectors of linearized gravitational waves propagating on a Schwarzschild background. In contrast to the case of normal modes, there may be multiple essentially isospectral partners, each missing a different state. The SUSY transformation preserves orthonormality under a bilinear map which is the analog of the usual inner product for conservative systems. SUSY transformations can lead to doubled quasinormal and total-transmission modes; this phenomenon is analysed and illustrated. The existence or otherwise of SUSY partners is also relevant to the question of *inversion*: are open wave systems uniquely determined by their complex spectra?

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## I. INTRODUCTION

### A. Supersymmetric quantum mechanics

In this paper we consider supersymmetry (SUSY) in the one-dimensional Klein–Gordon equation (KGE)

$$[\partial_t^2 - \partial_x^2 + V(x)] \phi(x, t) = 0 \quad , \quad (1.1)$$

and especially in the corresponding eigenvalue problem

$$H\phi_n(x) = \omega_n^2 \phi_n(x) \quad , \quad (1.2)$$

where

$$H = -\partial_x^2 + V(x) \quad . \quad (1.3)$$

The boundary conditions will be specified later. In so far as the interest centers on the time-independent problem (1.2) and the spectrum, the case of the Schrödinger equation, to which reference is usually made, is included if the eigenvalues are simply relabeled by  $\omega^2 \mapsto \omega$ .

If there exists another system described by

$$\tilde{H} = -\partial_x^2 + \tilde{V}(x) \quad , \quad (1.4)$$

such that  $H$  and  $\tilde{H}$  have the same spectrum (or the same spectrum apart from one state), and moreover if the states in the two systems are related by

$$\tilde{\phi}(x) = A\phi(x) \quad , \quad (1.5)$$

where

$$\begin{aligned} A &= \partial_x + W(x) \quad , \\ -A^\dagger &= \partial_x - W(x) \quad , \end{aligned} \quad (1.6)$$

then the two systems are said to be SUSY partners [1]. The term “super” originates from boson–fermion symmetry in field theory, but here simply refers to a relationship between the states of two different systems, whereas symmetry relates states of the same system. SUSY for quantum mechanics is usually discussed for *normal modes* (NMs), i.e., solutions to (1.1) and (1.2) which vanish at spatial infinity and which are square-integrable [2]. The results for NMs are well known [3] and will be covered in the discussion below.

In order for (1.5) to preserve the spectrum, one needs

$$AH = \tilde{H}A \quad . \quad (1.7)$$

Our task is simply to analyse this condition when applied to open systems.

## B. Quasinormal modes and total-transmission modes

In open systems, waves are not confined, but can escape to infinity: acoustic waves escape from a musical instrument; electromagnetic waves escape from a laser; and linearized gravitational waves propagating on a Schwarzschild background escape to infinity and into the horizon. These systems are often described (e.g., in the case of gravitational waves [4]) by the KGE (1.1), or (e.g., in the case of optics [5]) by the wave equation

$$[\rho(x)\partial_t^2 - \partial_x^2]\phi(x, t) = 0 \quad , \quad (1.8)$$

to which the KGE can be transformed [6]. This transformation has certain subtleties when there are bound states in the KGE; see Appendix A.

We assume for the moment that  $V$  (or in the case of the wave equation  $\rho - 1$ ) has finite support. This assumption is natural for describing a system in the finite parts of space, surrounded by a trivial medium such as vacuum, and it leads to a simple formalism. However, some of the superpartners of finitely supported potentials, as well as the examples of Sections V and VI, prompt the study of SUSY under the weaker condition  $V(x) \rightarrow 0$  as  $x \rightarrow \pm\infty$ . The analysis then becomes considerably more subtle, and is therefore relegated to Appendix B.

We shall consider first of all states satisfying the so-called outgoing-wave condition, which is most easily stated for an eigenfunction of frequency  $\omega$ :

$$\phi(x) \propto e^{\pm i\omega x} \quad , \quad x \rightarrow \pm\infty \quad , \quad (1.9)$$

or equivalently

$$\frac{\phi'(x)}{\phi(x)} = \pm i\omega \quad , \quad x \rightarrow \pm\infty \quad . \quad (1.10)$$

Under these boundary conditions, discrete eigenvalues fall into two classes. First, there could be bound states or NMs [7]; because  $V$  vanishes at infinity, these must (from the Schrödinger point of view) have a negative energy  $\omega^2$ , and hence  $\omega$  is purely imaginary. According to the condition (1.9), we put  $\text{Im}\omega > 0$ . Second, there could be quasinormal modes (QNMs) with complex eigenvalues  $\omega^2$ . Because these waves (in contrast to the bound states) are genuinely outgoing, probability (or energy) in the finite parts of space decreases, so  $\text{Im}\omega < 0$  [8]. We here assume that the loss is only due to the escape of the waves, i.e., only through the boundary conditions. In particular, the potential  $V$  is real. (Absorption may be described by a complex  $V$ , but causality then requires dispersion; the necessary generalization of (1.1) [9] will not be discussed here.)

The complex QNM frequencies are often directly observable: for example the central frequency and width of an optical line observed from a laser cavity, or the rates of repetition and decay of a gravitational-wave signal that may within the next decade be detected by instruments such as LIGO [10]. By way of orientation, Figure 1a shows a square well (solid line, ignore broken line), and Figure 1b shows, on the complex  $\omega$ -plane, the distribution of NM and QNM frequencies (crosses and circle, ignore triangle). Several features may be noticed. (a) Bound states or NMs [2] are shown in the upper half-plane; in this example there are three. (b) QNMs have  $\text{Im}\omega < 0$ , and are in the lower half-plane. Provided  $\text{Re}\omega \neq 0$ , they occur in pairs:  $\omega_{-n} = -\omega_n^*$ , as is readily shown by conjugating the defining equation and boundary conditions. (c) There may be QNMs, not paired, which have  $\text{Re}\omega = 0$ ; in this example there is one. These *zero modes* [11] will be of particular importance below.

Even though the QNM eigenfunctions are not square-integrable and do not form a Hilbert space (at least not in the conventional sense), they are nevertheless useful for analyzing outgoing waves. As stated above, QNM frequencies are directly observable, and as illustrated by the example in Figure 1, the spectrum is typically much richer than that

for NMs. In fact, the spectrum is so rich that, under some broad conditions, namely that  $V(x)$  vanishes identically outside an interval  $[-a, a]$  and has a singularity at  $x = \pm a$ , the QNMs together with any possible NMs are *complete* in at least part of spacetime [12], so that the wave signal observed at a point  $x$  can be represented as

$$\phi(x, t) = \sum_n a_n \phi_n(x) e^{-i\omega_n t} \quad , \quad (1.11)$$

where the sum is over all QNMs (including zero modes) and NMs [13]. In cases where the QNMs are not complete, it may still be possible to characterize the remainder, which could be, for example, a power law in  $t$  [14,15]. Moreover, when the QNMs are complete, one can set up a formalism that almost completely parallels the case of NMs [6,16]: under a bilinear map which is the analog of the usual inner product (but which is linear in both vectors rather than linear in one and conjugate linear in the other), the Hamiltonian turns out to be symmetric (the analog of self-adjoint). This allows much of the well-known mathematical formalism developed for NMs to be transcribed. The mathematical structure can also be written elegantly using a biorthogonal formalism [17]. One can even second-quantize using these QNMs as a basis (e.g., to discuss photon creation and annihilation in an optical cavity) [18]. These developments have been reviewed in Ref. [19]. An important point of this paper is that much of the mathematical structure is preserved under SUSY (Section VII).

It is straightforward and indeed natural to include total-transmission modes (TTMs) in the discussion as well; these are defined by the incoming-wave condition at one extreme and the outgoing-wave condition at the other. The TTMs propagating from the left (TTM<sub>L</sub>) and those propagating from the right (TTM<sub>R</sub>) satisfy

$$\phi(x) \propto e^{\pm i\omega x} \quad , \quad |x| \rightarrow \infty \quad , \quad (1.12)$$

with the  $+$  ( $-$ ) sign for a TTM<sub>L</sub> (TTM<sub>R</sub>).

Given the striking parallel between NMs and QNMs, and especially since the QNM spectrum is often richer than the NM spectrum, it is logical to ask whether the entire concept of SUSY can be generalized to QNMs: namely, given a real potential  $V$ , can one find a real partner potential  $\tilde{V}$  such that the outgoing waves for the two have the same set of complex eigenfrequencies (or the same set apart from one state) and are related by (1.5)? The same question can be asked of TTMs. This paper sets out to answer this question, which turns out *not* to be more difficult, even though one has to match a set of *complex* rather than real frequencies. Indeed, when NMs and QNMs are studied together, the relationship between the spectra of  $H$  and  $\tilde{H}$  becomes clearer.

Relations between such pairs of open systems have been discussed in terms of the transmission amplitude  $T(\omega)$  and the reflection amplitude  $R(\omega)$  etc., for solutions at definite frequencies. However, an *operator* SUSY transformation such as (1.5) is more powerful: given a  $\phi(x, t)$  that solves the dynamics of  $H$ , one can immediately generate the corresponding solution  $\tilde{\phi}(x, t) = A\phi(x, t)$  which solves the dynamics of  $\tilde{H}$ , *without* having to first project  $\phi$  onto frequency eigenfunctions.

SUSY is closely related to the question of *inversion*: given the complete set of complex QNM frequencies, can  $V(x)$  be determined uniquely? Obviously, if isospectral SUSY partners exist, then unique inversion would not be possible. The converse need not be the case, since there could be isospectral systems not related by a SUSY transformation such as (1.5). It may be well to recall the classic results for NMs [20]: for a finite interval, say  $[0, a]$ , if the (real) spectrum is given for the *two* dynamical systems defined respectively by (a)  $\phi(0) = \phi(a) = 0$  (the nodal system), and (b)  $\phi'(0) = \phi(a) = 0$  (the antinodal system), then the potential can be uniquely determined. (Equivalently, one can extend the problem to  $[-a, a]$  and seek a *symmetric* potential  $V(x)$  given the spectrum for (a) the even-parity sector and (b) the odd-parity sector.) Effective numerical algorithms have recently been given [21,22]. Since the QNMs of open systems are complex and carry twice the amount of information, it is tempting to speculate that the frequencies for *one* set of boundary conditions (e.g., outgoing waves at both ends of a finite interval) may be enough for unique inversion. There is some numerical evidence to support this conjecture, at least in some cases [23,24].

The possibility of inversion from *one* spectrum is intriguing, since physically all the information one has is the time-dependent signal and hence in principle the set of complex frequencies under *one* set of boundary conditions [25]. For example, without wishing to belittle the technical difficulties involved, one may imagine that the signal received by LIGO could yield the complex eigenvalues and from these one could determine  $V(x)$ , which describes the background curvature. This could then, in principle, provide a novel astrophysical tool — using gravitational waves to probe the intervening spacetime between the source and the observer. It has been suggested that the QNM spectrum may be a definitive signature of black holes [26]. Moreover, a toy model has been given for the corresponding forward problem of perturbing a black hole: if the hole is “dirty”, i.e., perturbed by surrounding matter, how would this be revealed in the shifts of the complex frequencies [27]? The inverse of this perturbative problem would in principle

allow the nature of the astrophysical environment around a black hole to be revealed through the shifts in the QNM spectrum.

With these motivations, in this paper we study SUSY for QNMs and TTMs. The formalism is given in Section II, and is in fact almost the same as that for NMs [28]. SUSY transformations can be classified into three discrete types plus one continuous type. Examples for square wells and barriers are given in Section III. It is found that the familiar transformations not only establish an equivalence between the NM spectra of partner potentials, but preserve the QNM spectrum as well. Moreover, by considering NMs and QNMs together, a clearer picture emerges. The reflection and transmission amplitudes for SUSY partners are related in a simple way, as is shown in Section IV. Section V deals with the Pöschl–Teller (PT) potential, which has a number of interesting features: (a) for a range of parameters, the PT potential has an infinite number of zero modes, and many of these allow corresponding SUSY transformations, leading to infinitely many essentially isospectral partners; (b) two of these transformations lead to partner potentials also of the PT type, but with different amplitudes; (c) thus the partner can again be transformed to yet other PT systems, *ad infinitum*; and (d) a subset of PT potentials are SUSY-equivalent to the free field, and therefore have total transmission at all positive energies. In Section VI, the Regge–Wheeler (RW) and Zerilli potentials, which respectively describe the axial and polar sectors of linearized perturbations of a Schwarzschild spacetime, are shown to be SUSY partners. This result is not novel, but is here cast into the general framework of SUSY. The orthonormality of QNMs under SUSY is presented in Section VII, in which SUSY is generalized to the two-component formalism appropriate to open systems. SUSY can also lead to *doubled* QNMs and TTMs, in the sense that two such modes occur at the same frequency. This phenomenon, which has no counterpart in conservative systems, is analysed and illustrated in Section VIII. Finally, a discussion is given in Section IX.

## II. FORMALISM

It is readily shown, without reference to any boundary conditions and hence with equal validity for NMs, QNMs and TTMs, that  $H$  and  $\tilde{H}$  have the same spectrum (with the possible exception of one state, to be discussed below), provided (1.7) holds. It is readily shown that the two potentials can be written as

$$\begin{aligned} V(x) &= W(x)^2 - W'(x) + \Omega^2 \quad , \\ \tilde{V}(x) &= W(x)^2 + W'(x) + \Omega^2 \quad , \end{aligned} \tag{2.1}$$

with  $W(x)$  (called the SUSY potential) as in (1.6) and for some constant  $\Omega^2$ . Moreover, the Hamiltonians can be represented as

$$\begin{aligned} H &= A^\dagger A + \Omega^2 \quad , \\ \tilde{H} &= A A^\dagger + \Omega^2 \quad . \end{aligned} \tag{2.2}$$

It follows that if  $\phi(x)$  is an eigenfunction of  $H$ , then the partner function  $\tilde{\phi}(x) = A\phi(x)$  (provided it does not vanish) is an eigenfunction of  $\tilde{H}$  with the same eigenvalue. The two partner systems can be put into one linear space by introducing Pauli spinors, with  $H$  and  $\tilde{H}$  associated with  $1 \pm \sigma_z$  and  $A, A^\dagger$  associated with  $\sigma_\pm$ . This transcription of the usual formalism remains valid even when the eigenvalues are complex. We defer all discussions of normalization to Section VII.

Upon reversing the sign of  $W$ , (a)  $V$  and  $\tilde{V}$  are interchanged (see (2.1)), and (b)  $A$  and  $-A^\dagger$  are interchanged (see (1.6)); thus the inverse mapping from  $\tilde{H}$  back to  $H$  is (up to a sign) achieved by  $A^\dagger$ .

We may regard (2.1) as a Riccati equation for  $W$  in terms of the given  $V$ . As  $V(x)$  is assumed to vanish for  $x \rightarrow \pm\infty$ , we have  $W^2 = -\Omega^2$  as  $x \rightarrow \pm\infty$ . The first-order Riccati equation can satisfy two boundary conditions only at special values of  $\Omega^2$ ; this condition becomes familiar if we define the *SUSY generator*  $\Phi(x)$  by

$$W(x) = -\frac{\Phi'(x)}{\Phi(x)} \quad . \tag{2.3}$$

Then (2.1) implies that  $\Phi$  must be an eigenfunction of  $H$  with eigenvalue  $\Omega^2$ :

$$H\Phi(x) = \Omega^2\Phi(x) \quad . \tag{2.4}$$

The various SUSY transformations are then related, in a one-to-one manner, to solutions of (2.4). Before proceeding further, we should note that even for the QNM case with complex frequencies, since  $\tilde{V}$  is required to be real,  $W$  and hence  $\Phi$  and  $\Omega^2$  also have to be real (in the case of  $\Phi$  up to an irrelevant overall phase) [29].

The task is therefore to analyze the boundary conditions. First, suppose  $\Omega^2 > 0$ , so that  $\Omega$  is real. Then outside the support of  $V$ ,  $\Phi$  is oscillatory: either complex (e.g.  $e^{i\Omega x}$ ), inadmissible since it leads to a complex  $W$ ; or real (e.g.  $\sin \Omega x$ ), inadmissible since its nodes lead to singularities in  $W$ . Thus,  $\Omega^2 \leq 0$ , and we denote  $K \equiv |\Omega|$ . The somewhat special marginal case  $\Omega = 0$  is dealt with in Appendix B 6.

At each spatial extreme ( $x \rightarrow \pm\infty$ ), the solution can be either (a) purely decreasing ( $\Phi \propto e^{-K|x|}$ ), denoted as D; (b) purely increasing ( $\Phi \propto e^{K|x|}$ ), denoted as I; or (c) a mixture of the two terms, i.e.,

$$\Phi(x) = ce^{K|x|} + de^{-K|x|} \quad , \quad (2.5)$$

with  $c, d \neq 0$ , to be denoted as M. Considering both extremes, the possible solutions for  $\Phi$  can be classified into the following types, in obvious notation:

$$\begin{aligned} \text{Type 1} &= \text{DD} \quad , \\ \text{Type 2} &= \text{II} \quad , \\ \text{Type 3} &= \text{DI, ID} \quad , \\ \text{Type 4} &= \text{MM, MD, MI, DM, IM} \quad . \end{aligned} \quad (2.6)$$

The first three types can only occur at discrete values of  $\Omega$ , and can be characterized as follows. A solution decreasing at both ends is an NM (DD = NM); we associate these with waves going as  $e^{i\Omega|x|}$ , with  $\Omega = iK$ . A solution which is purely increasing at both ends is a QNM (II = QNM); we again associate these with waves going as  $e^{i\Omega|x|}$ , but now with  $\Omega = -iK$ . A DI solution goes as  $e^{Kx}$  at both extremes ( $K > 0$ ). We can associate this with either (a) the time-dependent solution  $e^{i\Omega x}e^{-i\Omega t}$  with  $\Omega = -iK$ , namely a TTM<sub>L</sub>, or (b) the time-dependent solution  $e^{-i\Omega' x}e^{-i\Omega' t}$  with  $\Omega' = iK$ , namely a TTM<sub>R</sub>. Purely as a matter of convention, we shall only discuss states in the lower half-plane (in this case the TTM<sub>L</sub>), always noting that there is a TTM propagating in the opposite direction at the opposite value of the frequency which is not shown. Thus, with this convention, DI = TTM<sub>L</sub>; similarly ID = TTM<sub>R</sub>. This accords with the convention in discussions of black-hole modes. We shall in the rest of this section pay particular attention to the disappearance and appearance of the four classes of solutions (NM, QNM, TTM<sub>L</sub>, TTM<sub>R</sub>) under SUSY.

In contrast, Type 4 solutions are allowed for continuous  $\Omega$ . There is an additional difference. For Types 1 to 3, the logarithmic derivative outside  $I$  is exactly  $\pm K$ , so  $W' = 0$ . Then, since  $V$  vanishes outside  $I$ , so does  $\tilde{V}$ . However, for Type 4, the logarithmic derivative is given by (e.g., for  $x \rightarrow +\infty$ )

$$W = -K + \frac{2dK}{c}e^{-2Kx} + \dots \quad , \quad (2.7)$$

so that  $\tilde{V}$  would have an exponential tail, taking us outside the simplified theory presented in this section.

We should check immediately that outgoing/incoming boundary conditions are preserved under SUSY. To do so, let  $\phi$  be, for example, an outgoing wave,  $\phi(x) = Ce^{i\omega x}$  as  $x \rightarrow \infty$ ; then operating with  $A$ , we find

$$\begin{aligned} \tilde{\phi}(x) &= [\partial_x + W(x)]\phi(x) \\ &= (i\omega + W_+)Ce^{i\omega x} \quad , \end{aligned} \quad (2.8)$$

where we have introduced the constants

$$W(x \rightarrow \pm\infty) = W_{\pm} \quad , \quad (2.9)$$

which, depending on which type is being discussed, would equal  $\pm K$ . Thus  $\phi$  and  $\tilde{\phi}$  always satisfy the same type of boundary conditions.

Two properties of the generator  $\Phi$  ought to be mentioned. First, it is annihilated by the operator  $A$ :  $A\Phi = 0$ ; this follows trivially from (1.6) and (2.3). Second, from (2.1) we see that the partner systems are related by reversing the sign of  $W$ , so in view of (2.3), the corresponding generator for the inverse transformation from  $\tilde{H}$  to  $H$  is  $\tilde{\Phi} = \Phi^{-1}$ ; this function is guaranteed to be an eigenfunction of  $\tilde{H}$  also with the eigenvalue  $\Omega^2$ . (Despite the notation,  $\tilde{\Phi}$  is not the SUSY partner of  $\Phi$ :  $\tilde{\Phi} \neq A\Phi$ .) Because of the reciprocal relation, the boundary conditions for  $\tilde{\Phi}$  are obvious: under  $\Phi \mapsto \tilde{\Phi}$ , D  $\mapsto$  I, I  $\mapsto$  D, and M  $\mapsto$  D.

Thus, under SUSY, one state  $\Phi$  disappears and an extra state  $\tilde{\Phi}$  appears, with boundary conditions (e.g., DD, II) readily deduced from the above description. In view of the association of these boundary conditions with the four types of solutions (e.g., DD = NM, II = QNM), we can summarize the changes in the number of these solutions as

in Table 1. In this table, the entries  $\pm 1$  in the “NM” column refer to transitions  $0 \leftrightarrow 1$  only, because there can be no doubled bound states (cf. at the end of the Introduction) in 1-d systems; see also Appendix B 5.

In particular, if  $\Phi$  is itself an NM or QNM (Type 1 or 2), then in the partner system one state at  $\Omega = \pm iK$  disappears and another state at  $-\Omega = \mp iK$  appears instead. In these cases, the systems, differing by one state whose frequency is reversed, are said to be *essentially isospectral*, and the mapping  $A$  between them constitutes a *good* SUSY transformation [3]. Where  $\Phi$  is not an NM or QNM of  $H$ , and  $\tilde{\Phi}$  is not an NM or QNM of  $\tilde{H}$  (Type 3), then the two systems are *strictly isospectral* (for both the real NM and the complex QNM spectra), corresponding to *broken* SUSY [30].

Some of these properties are of course well known; but the more systematic perspective is possible only if attention is paid to the QNMs (and TTMs) as well. These general remarks will next be illustrated with examples.

### III. SQUARE WELLS AND BARRIERS

In this section we illustrate the different types of SUSY transformations using square wells and barriers for  $V$ ; this has the advantage that all different cases can be solved analytically. All examples are presented graphically with a uniform format. The top diagrams show the original potential  $V$  (solid line) and the partner potential  $\tilde{V}$  (broken line). The bottom diagrams show the complex  $\omega$ -plane. The NMs and QNMs common to both systems are shown with crosses. The mode corresponding to  $\Phi$  ( $\tilde{\Phi}$ ) is shown with a circle (triangle); these are present in one system but not in the other. In each case, we have verified the lowest few QNMs of  $\tilde{H}$  numerically.

#### A. Type 1: $\Phi$ is a normal mode

The Type 1 solution, denoted as DD, corresponds to  $\Phi$  itself being a bound state or NM. This occurs only if  $V$  is sufficiently attractive. No matter how many bound states there are, only the ground state is acceptable as the generator  $\Phi$ , since the others would have nodes leading to a singular  $W$  — exactly as in the familiar discussion of SUSY between NM spectra. In this case, the two spectra (both NMs and QNMs) are identical except that: (a)  $\tilde{H}$  has one fewer NM, since the ground state in  $H$ , namely  $\phi_0 = \Phi$ , is annihilated by  $A$ ; (b)  $\tilde{H}$  has one extra QNM, since the corresponding function  $\tilde{\Phi} = \Phi^{-1}$  is a solution with eigenvalue  $\Omega^2$  but which is increasing (i.e., is a purely outgoing wave) for  $x \rightarrow \pm\infty$ . The systems are essentially isospectral.

As an example, take  $V$  to be a square well (Figure 1):

$$V(x) = V_0 \theta(a - |x|) \quad , \quad (3.1)$$

where  $\theta$  is the unit step function, and with parameters say  $V_0 = -20$ ,  $a = 1$ . The even wavefunctions, for example, are given by

$$\phi(x) = A \cosh \alpha x \quad , \quad (3.2)$$

for  $|x| < a$ , where

$$-\alpha^2 + V_0 = \omega^2 \quad . \quad (3.3)$$

The logarithmic derivative at  $x = a$  must match onto an outgoing wave:

$$\alpha \tanh \alpha a = i\omega \quad , \quad (3.4)$$

where the case of NMs is included if we take  $\text{Re } \omega = 0$ ,  $\text{Im } \omega > 0$ . The odd sector can be treated similarly.

For this choice of parameters, there are 3 bound states, at  $\omega = 4.28i, 3.68i, 2.47i$ . Taking the lowest of these as the auxiliary function  $\Phi$  (so that  $\Omega = +iK$  with  $K = 4.28$ ), we find for  $|x| < a$

$$\begin{aligned} \Phi(x) &= A \cos qx \quad , \\ W(x) &= q \tan qx \quad , \\ \tilde{V}(x) &= V_0 + 2q^2 \sec^2 qx \quad , \end{aligned} \quad (3.5)$$

where  $q = \sqrt{-V_0 - K^2}$  is real; with the present parameters  $q = 1.28$ . Figure 1a shows the two partner potentials and Figure 1b compares the spectra. The NM in  $H$  but not  $\tilde{H}$  is shown by the circle at  $\omega = iK$ ; the QNM in  $\tilde{H}$  but not  $H$  is shown by the triangle at  $\omega = -iK$ .

### B. Type 2: $\Phi$ is a quasinormal mode

The Type 2 solution, denoted as II, corresponds to  $\Phi$  itself being a QNM. Since  $\Omega^2$  has to be real, the frequency is purely imaginary and  $\Phi$  is a zero mode. We note two features. First, not all potentials will have a zero mode. Second, in distinct contrast to the NM case, there may be *several* nodeless QNM zero modes, all of which are candidates for the generator  $\Phi$ .

To illustrate the first property, consider the example of a square barrier, i.e., (3.1) with  $V_0 > 0$ . Again we take  $a = 1$ , and allow  $V_0$  to vary. The formulas are similar to (3.1)–(3.5) with the sign of  $V_0$  reversed, and will not be shown. In the even sector there are two zero modes for small values of  $V_0$ ; as  $V_0$  increases, these move closer together and merge at  $V_0 = V_c = 0.291$  [31]; for higher barriers there are no zero modes. There are no zero modes in the odd sector, and in any event an odd state is not eligible as a generator (because there is a node). To be definite, we take  $V_0 = 0.16$ , in which case the zero modes are at  $\omega = -0.181i, -2.500i$ , with  $\alpha = 0.242, 2.506$  in (3.2). These wavefunctions, being  $\cosh \alpha x$  within  $[-a, a]$  and a real exponential for  $|x| > a$ , clearly have no nodes.

This takes us to the second property alluded to above, namely, that the existence of more than one nodeless QNM is by no means exceptional; some general remarks are given in Appendix C. Multiple nodeless QNMs allow different SUSY transformations. Continuing with the example, we show separately the case where the state at  $\Omega = \omega_1 = -0.181i$  is chosen to be  $\Phi$  (Figure 2) and the case where the state at  $\omega_2 = -2.500i$  is chosen (Figure 3). The two partner potentials  $\tilde{V}$  are different. In each case, one QNM has disappeared and an extra NM has emerged. The systems are again essentially isospectral.

### C. Type 3: $\Phi$ is a total-transmission mode

To be definite, we take the Type 3 solution to be DI:  $\Phi$  is decreasing as  $x \rightarrow -\infty$  and increasing as  $x \rightarrow \infty$ , i.e., a TTM<sub>L</sub> according to the convention adopted. A simple example can be provided by a multi-step square barrier:

$$V(x) = \begin{cases} V_1 & \text{if } |x| \leq b \\ V_0 & \text{if } b < |x| \leq a \\ 0 & \text{if } |x| > a \end{cases} . \quad (3.6)$$

We take  $b = 0.1, a = 1.0, V_1 = -10.0, V_0 = 1.0$ . A TTM<sub>L</sub> is found for  $\Omega = -0.990i$ , and it can be used to generate a strictly isospectral SUSY partner: all the NMs and QNMs are preserved. Figure 4a shows the partner potential, and Figure 4b shows the QNM distribution. Figure 5a shows the generator, while Figures 5b and 5c respectively show the NM  $\tilde{\phi}_1$  and the QNM  $\tilde{\phi}_2$  in the partner system  $\tilde{V}$ , with eigenvalues  $\omega_1 = 0.498i$  and  $\omega_2 = -1.570i$ . (Because in this example  $V$  is symmetric, there is a TTM<sub>R</sub> at the same  $\Omega$ . However, that mode would generate a different  $\tilde{V}$ , in fact the parity-partner of the one considered here.)

### D. Type 4: Continuous transformations

As an example, consider the same square barrier as in Section IIIB, i.e.,  $V_0 = 0.16, a = 1.0$ . There are many sub-categories for Type 4 transformations (cf. (2.6)). We choose symmetric boundary conditions, i.e., (2.5) with the same values of  $c$  and  $d$  for both  $x \rightarrow \pm\infty$ . Within the range  $|x| \leq a$ ,  $\tilde{V}$  is again given by (3.5). However, for  $|x| > a$ , there is an exponential tail:

$$\tilde{V}(|x| > a) = 2K^2 \left[ 1 - \left( \frac{ce^{K|x|} - de^{-K|x|}}{ce^{K|x|} + de^{-K|x|}} \right)^2 \right] . \quad (3.7)$$

Turning to the mode structure, it turns out that  $\tilde{\Phi}$  is “universal” in that it is an NM, a QNM, and a TTM<sub>L/R</sub> at the same time, because outgoing and incoming waves coincide in the partner at  $\omega = -iK$ . (This somewhat surprising phenomenon is explained in Appendix B.) As in other cases, modes at eigenvalues  $\omega^2 \neq -K^2$  are not affected by SUSY. Figure 6 shows an example, where we have chosen  $K = 3.0$ , implying  $d/c = -0.829$ .

## IV. REFLECTION AND TRANSMISSION AMPLITUDES

The relation between the reflection and transmission amplitudes of SUSY partners is known in general [3], and has also been used in the specific context of gravitational waves propagating on a black-hole background [4]. Here we record the results for completeness.

The reflection and transmission amplitudes (for a wave incoming from the left) are defined by

$$\phi(x) = \begin{cases} e^{i\omega x} + R_L(\omega)e^{-i\omega x} \\ T_L(\omega)e^{i\omega x} \end{cases}, \quad (4.1)$$

for  $x \rightarrow -\infty$  and  $x \rightarrow \infty$  respectively. Now applying  $A = \partial_x + W$  and recognizing that  $W(x)$  for large  $x$  equals finite constants (2.9), the corresponding wavefunction in the partner system is

$$\tilde{\phi}(x) = \begin{cases} (i\omega + W_-)e^{i\omega x} + (-i\omega + W_-)R_L(\omega)e^{-i\omega x} \\ (i\omega + W_+)T_L(\omega)e^{i\omega x} \end{cases} \quad (4.2)$$

respectively for  $x \rightarrow \mp\infty$ . Normalizing the incoming wave on the left to unity, we find the corresponding amplitudes in the SUSY partner to be

$$\begin{aligned} \tilde{R}_L(\omega) &= \frac{-i\omega + W_-}{i\omega + W_-} R_L(\omega) \quad , \\ \tilde{T}_L(\omega) &= \frac{i\omega + W_+}{i\omega + W_-} T_L(\omega) \quad . \end{aligned} \quad (4.3)$$

The constants  $W_+$  and  $W_-$  are real and always equal in magnitude. For real  $\omega$  it then follows that  $|\tilde{R}_L(\omega)|^2 = |R_L(\omega)|^2$ ,  $|\tilde{T}_L(\omega)|^2 = |T_L(\omega)|^2$ . In the case of Type 3 SUSY transformations,  $W_- = W_+$  and one has the stronger condition  $\tilde{T}_L(\omega) = T_L(\omega)$ .

Similar relations can be obtained for the amplitudes  $R_R$  and  $T_R$  defined for a wave incoming from the right; it suffices to change  $W_{\pm} \rightarrow W_{\mp}$  and  $\omega \rightarrow -\omega$  in the prefactors above. In fact, one has the relations

$$\begin{aligned} R_R(\omega) &= -\frac{T_L(\omega)R_L(-\omega)}{T_L(-\omega)} \quad , \\ T_R(\omega) &= T_L(\omega) \quad . \end{aligned} \quad (4.4)$$

Although the transmission and reflection amplitudes for pairs of related open systems are useful, they relate to states of definite frequency, and are less general than the operator SUSY transformation.

## V. PÖSCHL–TELLER POTENTIALS

### A. General properties

We next illustrate with two examples related to the PT potential [32]:

$$V(x) = \mathcal{V} \operatorname{sech}^2(x/b) \quad , \quad (5.1)$$

which has an exponential tail:  $V(x) \sim \mathcal{V}e^{-2|x|/b}$ . This means that the theory of Section II is not strictly sufficient (cf. Appendix B), but for simplicity we shall gloss over the complications as much as possible, only pointing them out where appropriate. The PT potential admits analytic solutions, and because of its exponential tail is sometimes used as a proxy to illustrate some properties of the effective potentials that arise in the study of gravitational waves; the latter also have exponential tails for  $x \rightarrow -\infty$ , where  $x$  is the tortoise coordinate and  $-\infty$  corresponds to the event horizon. (See Section VI for details.) The PT potential is especially interesting from the point of SUSY, in at least three ways.

(a) There is a class of SUSY transformations (Types 1 and 2) which map one PT potential to another of the same width ( $\mathcal{V} \mapsto \tilde{\mathcal{V}}$ ,  $b \mapsto b$ ).

(b) There are Type 4 transformations which map the free field ( $\mathcal{V} = 0$ ) to a PT potential with  $b^2\mathcal{V} = -2$ .

(c) By again transforming this potential as in (a), one obtains a string of other PT potentials with  $b^2\mathcal{V} = -\ell(\ell+1)$ , with  $\ell$  being an integer.

These results are interesting because (a) implies that PT potentials are shape-invariant under SUSY, while (b) and (c) show that certain PT potentials are SUSY-equivalent to a free field, and therefore have total transmission for *all* positive energies. Attractive PT potentials have a finite number of bound states, while repulsive PT potentials obviously have none; thus the existing SUSY literature, which focuses on NMs, has not given a systematic account of these properties.

To pave the way for discussing SUSY, we first summarize the known results for the PT potential. For any  $\omega$ , the wavefunction that is outgoing at  $x \rightarrow -\infty$  reads

$$\begin{aligned} \phi(\omega, x) &= [\xi(1 - \xi)]^{-i\omega b/2} \\ &\times {}_2F_1\left(\frac{1}{2} + q - i\omega b, \frac{1}{2} - q - i\omega b; 1 - i\omega b; \xi\right) \quad , \end{aligned} \quad (5.2)$$

where  ${}_2F_1$  is the hypergeometric function, and the variable  $\xi$  is related to  $x$  by

$$\xi = \frac{1}{1 + e^{-2x/b}} \quad . \quad (5.3)$$

The parameter

$$q = \sqrt{\frac{1}{4} - b^2\mathcal{V}} \quad (5.4)$$

will be especially important below. For generic  $q$ , these solutions are outgoing at the following values of  $\omega$ :

$$\omega = \omega_n^\pm(q) \equiv -\frac{i}{b}\left(n + \frac{1}{2} \pm q\right) \quad , \quad (5.5)$$

for  $n = 0, 1, 2, \dots$ , which thus are the (Q)NM frequencies of the PT potential. The states have parity  $(-1)^n$ . However, if  $q = \frac{1}{2} + \ell$  ( $\ell = 0, 1, 2, \dots$ ), the modes are restricted to  $i\omega b = -\ell, -\ell + 1, \dots, \ell$ : apparently, for half-integer  $q$  the two strings of modes (5.5) annihilate each other where they coincide. (Moreover, for these  $q$  the remaining modes are of a special type, coined “universal” in Section III D; see further Appendix B.)

From (5.4), it is clear that the value  $b^2\mathcal{V}$  is important, and in fact the situation can be classified as follows.

(a) If  $\frac{1}{4} < b^2\mathcal{V}$  (so that  $q$  is imaginary), the QNMs are paired with  $\omega_n^- = -(\omega_n^+)^*$ . This case will not be discussed here, because there are no real SUSY generators.

(b) If  $0 < b^2\mathcal{V} < \frac{1}{4}$  (so that  $q$  is real and the potential is repulsive), these QNMs are all zero modes lying on the imaginary axis. This case is interesting because many of these QNMs can be used as the auxiliary function  $\Phi$ .

(c) If  $b^2\mathcal{V} = 0$ , one obtains the free field, whose SUSY partners will be discussed in Section V E.

(d) If  $b^2\mathcal{V} < 0$ , one obtains an attractive PT potential, and it will be seen that these could be SUSY partners of repulsive PT potentials.

(e) If  $b^2\mathcal{V} = -\ell(\ell + 1)$ , the attractive PT potential is SUSY-equivalent to a free field.

These cases will be discussed in detail below. In fact, some of them could be anticipated from the eigenvalue formula (5.5), which has the important property:

$$\omega_n^\pm(\tilde{q}) = \omega_n^\pm(q) \quad (5.6)$$

for integers  $n$  and  $\tilde{n}$  iff

$$\tilde{q} = q \pm (n - \tilde{n}) \quad . \quad (5.7)$$

Likewise

$$\omega_n^\mp(\tilde{q}) = \omega_n^\pm(q) \quad (5.8)$$

for integers  $n$  and  $\tilde{n}$  iff

$$\tilde{q} = -q \pm (\tilde{n} - n) \quad . \quad (5.9)$$

Secondly, although (5.4) implicitly defines  $q \geq 0$  (where it is real), negative values can be admitted as well, and

$$\omega_n^+(q) \equiv \omega_n^-(-q) \quad ; \quad (5.10)$$

these are just different labelings of the same state. So by allowing negative values of  $q$ , (5.8)–(5.9) could be subsumed under (5.6)–(5.7). The equivalence in spectrum implied by (5.6) and (5.8) (which is not strict isospectrality because  $n$  and  $\tilde{n}$  have to be non-negative) already suggests a SUSY relationship, which will be presented in the following.

## B. Results for truncated potential

Because of the subtleties when the potential has a tail, we first consider a slightly altered problem — the PT potential truncated at  $x = |a|$ :

$$V(x) = \mathcal{V} \text{sech}^2(x/b) \theta(a - |x|) \quad . \quad (5.11)$$

We take  $\mathcal{V} = 3/16$ ,  $b = 1$ , and  $a = 2$  (solid line in Figure 7a). The QNMs can be found by requiring the logarithmic derivative at  $x = \pm a$  to be  $\pm i\omega$ ; the resulting frequencies are shown in Figure 7b (crosses and circle only). The lowest QNMs are approximately the same as in the untruncated case, though the higher QNMs are altered beyond recognition [33], while many new QNMs appear because of the reflections at  $x = \pm a$ . The important point is that there are still two zero modes, in this instance at  $\omega = -0.224i, -1.301i$ . Either of these can be used as  $\Phi$ ; to be definite, we take the first. The SUSY transformation is thus of Type 2. The partner potential is shown in Figure 7a (broken line). We note two features: (a)  $\tilde{V}$  vanishes outside  $[-a, a]$ , and (b)  $\tilde{V}$  in this case is attractive. The overall spectrum is shown in Figure 7b. Notice, as before, the appearance of an extra NM accompanying the disappearance of the QNM.

## C. Results for full potential

We now return to the full PT potential, the eigenvalues for which are given by (5.5). For  $b^2\mathcal{V} < \frac{1}{4}$ , all the QNMs  $\omega_n^\pm$  are zero modes. For states with odd  $n$ , the node at  $x = 0$  prevents their use as the generator  $\Phi$ . If  $\mathcal{V} > 0$ , all the even- $n$  eigenfunctions are nodeless (Appendix C) and therefore eligible as the generator, which we denote as  $\Phi_n^\pm$ . This then leads to an infinite number of SUSY potentials  $W_n^\pm$ , and correspondingly an infinite number of partner potentials  $\tilde{V}_n^\pm$ . Since we start with a QNM, these refer to Type 2 SUSY. For attractive potentials, the construction goes through at least for  $n = 0$ :  $\Phi_0^-$  is the ground state and hence certainly nodeless, and  $\Phi_0^+$  is readily checked to have the same property. Some arithmetic leads to the following explicit expression:

$$\tilde{V}_n^\pm(x) = [\mathcal{V} + A_n^\pm - 2B_n^\pm C_n^\pm G_2(x) + 2(B_n^\pm)^2 G_1(x)^2 \text{sech}^2(x/b) + 4B_n^\pm G_1(x) \tanh(x/b)] \text{sech}^2(x/b) \quad , \quad (5.12)$$

where

$$\begin{aligned} F_{n,p}^\pm(x) &= {}_2F_1\left(\frac{1}{2} + q - i\omega_n^\pm b + p, \frac{1}{2} - q - i\omega_n^\pm b + p; 1 - i\omega_n^\pm b + p; \xi\right) \quad , \\ G_{n,p}^\pm(x) &= F_{n,p}^\pm(x)/F_{0,p}^\pm(x) \quad , \\ A_n^\pm &= (-2n - 1 \mp 2q)b^{-2} \quad , \\ B_n^\pm &= \frac{n(n \mp 2q)b^{-2}}{-2n + 1 \pm 2q} \quad , \\ C_n^\pm &= \frac{(n-1)(n-1 \mp 2q)}{3 - 2n \pm 2q} \quad . \end{aligned} \quad (5.13)$$

Figure 8 shows the PT potential  $V$  and some of its SUSY partners  $\tilde{V}_n^\pm$  obtained in this manner.

## D. The case $n = 0$

We omit a detailed discussion of all the SUSY partners, but those generated with the  $n = 0$  states as the generator deserve special attention. From (5.13), we see that  $B_0^\pm = 0$ , and consequently  $\tilde{V}_0^\pm(x)$  are again of the PT form, with the same width  $b$  but different amplitudes, given by

$$b^2\tilde{\mathcal{V}} = b^2\mathcal{V} - 1 \mp 2q = b^2\mathcal{V} - 1 \mp \sqrt{1 - 4b^2\mathcal{V}} \quad . \quad (5.14)$$

This can be expressed more succinctly in terms of  $q$ :

$$\tilde{q} = |1 \pm q| \quad . \quad (5.15)$$

The equivalence in spectrum can now be readily verified from (5.7) and (5.9). In fact, we can use the freedom in the labeling convention provided by (5.10) to rewrite (5.15) as  $\tilde{q} = q \pm 1$ .

Since the partner  $\tilde{V}(x)$  is another PT potential, we can apply these transformations again and again. The result is a chain:

$$\begin{aligned} \cdots &\leftrightarrow -m+q \leftrightarrow \cdots \leftrightarrow -1+q \leftrightarrow q \\ &\leftrightarrow 1+q \leftrightarrow \cdots \leftrightarrow m+q \leftrightarrow \cdots \end{aligned} \quad (5.16)$$

This shows that the repeated application of SUSY in this manner forms a group isomorphic to the integers. The PT potentials at both extremes of (5.16) become increasingly attractive, with more and more NMs.

The PT potential is self-replicating in the sense that (some of) its SUSY partners are the same potential with a different amplitude. In Appendix D we show that conversely this condition leads uniquely to the PT potential.

### E. SUSY partners of the free field

The free field  $V(x) = 0$  is a special case of the PT potential, corresponding to  $q = \frac{1}{2}$ . Apply a Type 4 transformation by using the generator

$$\Phi(x) = \cosh Kx \quad (5.17)$$

A simple calculation then gives

$$\tilde{V}(x) = -2K^2 \operatorname{sech}^2 Kx \quad (5.18)$$

which is of the form (5.1). We see that an attractive PT potential is a SUSY partner of the free field if

$$b^2\mathcal{V} = -2 \quad (5.19)$$

or equivalently  $q = \frac{3}{2}$ . This potential has one NM, which is just  $\tilde{\Phi} = \Phi^{-1}$ .

For half-integer  $q$  the QNM  $\omega_0^+$  is missing, see below (5.5). However, its limit as  $q \rightarrow \frac{1}{2} + \ell$ , being the even solution (cf. Section III D) at eigenvalue  $b^2\Omega^2 = -(\ell+1)^2$ , still generates a SUSY transform which can be verified to increase  $q$  by one. Successively applying these (Type 4) transformations one obtains increasingly attractive PT potentials, described by  $b^2\mathcal{V} = -2, -6, -12, \dots, -\ell(\ell+1), \dots$ ,  $q = \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \dots, \ell + \frac{1}{2}, \dots$ , with the number of NMs being respectively  $1, 2, 3, \dots, \ell, \dots$ .

These PT potentials are interesting because, being SUSY-equivalent to the free field, they obviously have total transmission for *all* complex frequencies. This property has recently been discussed in the context of electron transmission through semiconductor devices [34], but here it falls naturally into the SUSY framework.

### F. Coalescence of QNMs

The eigenvalue formula (5.4) implies that QNMs coalesce in two ways as the system parameters are tuned; these are quite distinct scenarios, and it is well to emphasize their difference.

First, as  $q \rightarrow 0$ , the QNMs  $\omega_n^+(q)$  and  $\omega_n^-(q)$  merge. At the point of coalescence, they become a double pole; this is related to an interesting Jordan-block structure in the evolution operator [31]. Beyond this point, the two QNMs separate in the real direction (i.e.,  $q$  becomes imaginary). The transition is exactly the same as a harmonic oscillator going through critical damping. This case will be further discussed in Section VIII. The same occurs for  $q \rightarrow n$ .

In contrast, as  $q \rightarrow \frac{1}{2}$ , the QNMs  $\omega_n^+(q)$  and  $\omega_{n+1}^-(q)$  merge. However, when they coalesce, the two poles disappear. With all poles eliminated, the dynamics become essentially trivial, with total transmission. The same occurs for  $q \rightarrow n + \frac{1}{2}$ .

## VI. REGGE–WHEELER AND ZERILLI POTENTIALS

An application of particular interest concerns linearized gravitational waves propagating on a Schwarzschild background. In each angular-momentum sector  $l \geq 2$  and for each spin  $s$  (for gravitational waves,  $s = 2$ ), these waves can be described by a scalar field  $\phi$  of the radial variable, satisfying a KGE with the potential  $V$  describing the nontrivial background [4]. However, the axial sector is described by the RW equation, which is (1.1) with the potential

$$V(x) = 2 \left( 1 - \frac{2m}{x_*} \right) \frac{x_*(n+1) - 3m}{x_*^3} , \quad (6.1)$$

while the polar sector is described by the Zerilli equation, which is (1.1) with the potential

$$\tilde{V}(x) = \left( 1 - \frac{2m}{x_*} \right) \times \frac{2n^2(n+1)x_*^3 + 6n^2mx_*^2 + 18nm^2x_* + 18m^3}{x_*^3(nx_* + 3m)^2} . \quad (6.2)$$

Here  $m$  is the black-hole mass, and  $x$  is the tortoise coordinate, related to the circumferential radius  $x_*$  by

$$x = x_* + 2m \log \left( \frac{x_*}{2m} - 1 \right) . \quad (6.3)$$

Note that  $x_* \in (2m, +\infty)$  maps to  $x \in (-\infty, +\infty)$ . The constant  $n$  is related to the angular momentum  $l$  by

$$n = \frac{1}{2}(l-1)(l+2) . \quad (6.4)$$

It has been noticed [35] that the two equations have the same QNM spectrum. For example, Figure 9 shows the distribution of QNM frequencies for  $l = s = 2$  for both equations. This can be understood because the QNMs of the two systems are related by

$$\tilde{\phi} = \left[ \frac{n(n+1)}{3m} + \frac{3m(x_* - 2m)}{x_*^2(nx_* + 3m)} \right] \phi + \frac{d}{dx} \phi , \quad (6.5)$$

which has been referred to as “intertwining” [36]. Eq. (6.5) is exactly a SUSY transformation, if we identify

$$W(x) = \frac{n(n+1)}{3m} + \frac{3m(x_* - 2m)}{x_*^2(nx_* + 3m)} . \quad (6.6)$$

The corresponding generator  $\Phi$  must be an eigenfunction of the RW equation with eigenvalue  $\Omega^2$ , where

$$\Omega = -i \frac{n(n+1)}{3m} . \quad (6.7)$$

This is exactly the so-called algebraically special frequency [37–39]. With the same eigenvalue,  $\tilde{\Phi} = \Phi^{-1}$  should be an eigenfunction of the Zerilli equation. Eq. (6.6) shows that  $W_+ = i\Omega$ ; hence,  $\Phi$  is incoming from the right. Since  $\Omega$  is not special for  $x \rightarrow \infty$  (cf. Appendix B; this can easily be verified using the Leaver series solution [40]), this already establishes that  $\Phi$  is not a QNM — as has sometimes been suggested [38]. However,  $W_- = i\Omega$  does not prove conclusively that it is outgoing to the left (cf. Appendix B 1), as implied by a conflicting suggestion that  $\Phi$  may be a TTM<sub>R</sub> [39]. In fact it can be shown that  $\Phi(x \rightarrow -\infty)$  is *not* outgoing, so that  $\Phi$  is not a TTM<sub>R</sub> and (6.5) is a Type 4 transform, being of category (b2) on the left in the terminology of Appendix B 3. However, these and other aspects will be reported separately.

## VII. ORTHONORMALITY

### A. Orthonormality for NMs

In the familiar discussion of SUSY for NMs, orthonormality can be preserved. There are two issues: (a) orthogonality is preserved because the transformed NMs are eigenvectors of the self-adjoint operator  $\tilde{H}$ ; and (b) normalization is preserved if the transformation is changed to

$$\phi_n \mapsto \tilde{\phi}_n = N_n \tilde{\phi}_n = N_n A \phi_n , \quad (7.1)$$

for each eigenstate  $n$  other than the ground state  $\Phi$ , with

$$N_n^{-2} = \frac{\langle \tilde{\phi}_n | \tilde{\phi}_n \rangle}{\langle \phi_n | \phi_n \rangle} . \quad (7.2)$$

It is readily shown that  $N_n^{-2} = \omega_n^2 - \Omega^2$ , a result that can also be read off as a special case of the derivation below for QNMs. Since  $\omega_n^2 - \Omega^2$  is the eigenvalue of the operator  $A^\dagger A$  (see (2.2)), the normalized SUSY transformation (7.1) can also be written in operator form

$$\phi \mapsto \underline{\tilde{\phi}} = A (A^\dagger A)^{-1/2} \phi \quad , \quad (7.3)$$

valid for any state  $\phi$ , not just frequency eigenfunctions. In this and similar formulas below, it will be understood that the domain is restricted to the subspace orthogonal to  $\Phi$ , on which therefore  $A^\dagger A$  is non-zero. This makes it formally easy to verify the preservation of inner products in the mapping from this subspace:

$$\begin{aligned} \langle \underline{\tilde{\psi}} | \underline{\tilde{\phi}} \rangle &= \langle \psi | (A^\dagger A)^{-1/2} A^\dagger A (A^\dagger A)^{-1/2} | \phi \rangle \\ &= \langle \psi | \phi \rangle \quad . \end{aligned} \quad (7.4)$$

However, when operating on a general wavefunction  $\phi$ , the factor  $(A^\dagger A)^{-1/2}$  can only be evaluated by projecting  $\phi$  onto the eigenfunctions, and scaling each component by  $N_n$ . Thus, in practice, the significant result is the evaluation of the factor  $N_n$ . We now generalize these concepts to QNMs.

## B. Normalization and inner product for QNMs

It is necessary to first digress and briefly review the concepts of orthogonality and normalization for QNMs. The central issue is that with the outgoing-wave boundary condition,  $H$  is *not* self-adjoint in the usual sense, and different QNMs are *not* orthogonal under the usual inner product. Likewise, the norm  $\int |\phi_n|^2 dx$  is divergent for a QNM, since the wavefunction grows exponentially at spatial infinity.

An appropriate normalizing factor for QNMs was first introduced by Zeldovich [41], and later generalized and applied to other situations [42], including models of linearized waves propagating on a background that is a Schwarzschild black hole plus a perturbation [27]:

$$\begin{aligned} (\phi_n, \phi_n) &= 2\omega_n \int_{-a}^a \phi_n(x)^2 dx \\ &\quad + i [\phi_n(-a)^2 + \phi_n(a)^2] \quad . \end{aligned} \quad (7.5)$$

This expression goes as  $\phi_n^2$  rather than  $|\phi_n|^2$ , and is in general not real. The upper limit of the integral and the first surface term can be shifted to any  $b_1 > a$  without affecting the value of (7.5); likewise, the lower limit and the second surface term can be shifted to any  $b_2 < -a$ . This expression is the correct normalizing factor in the sense that, e.g., under a perturbation  $V \mapsto V + \Delta V$ , the complex eigenvalues of the QNMs change by

$$\Delta(\omega_n^2) = \frac{\int \phi_n(x)^2 \Delta V(x) dx}{(\phi_n, \phi_n)} \quad . \quad (7.6)$$

Since one no longer has positivity, there is the possibility that  $(\phi_n, \phi_n) = 0$ . This exceptional case can be separately taken care of [31], and some interesting aspects are dealt with in Section VIII.

To go beyond the normalizing factor and discuss the analog of orthogonality, one has to first regard each state as a two-component vector  $\boldsymbol{\psi} = (\psi^1, \psi^2)^T \equiv (\psi, \partial_t \psi)^T$ , which is most easily motivated by noticing that the dynamics requires two sets of initial data. In terms of the two-component vector, one can define a bilinear map [6,16,19]

$$\begin{aligned} (\boldsymbol{\psi}, \boldsymbol{\phi}) &= i \left\{ \int_{-a}^a [\psi^1(x) \phi^2(x) + \psi^2(x) \phi^1(x)] dx \right. \\ &\quad \left. + [\psi^1(-a) \phi^1(-a) + \psi^1(a) \phi^1(a)] \right\} \quad . \end{aligned} \quad (7.7)$$

This will be seen to take the place of the usual inner product. Note that for an eigenfunction,  $(\phi_n, \phi_n)$  agrees with (7.5). The dynamics can be written in the first-order form  $i\partial_t \boldsymbol{\phi} = \mathcal{H} \boldsymbol{\phi}$ , with the  $2 \times 2$  evolution operator

$$\mathcal{H} = i \begin{pmatrix} 0 & 1 \\ \partial_x^2 - V & 0 \end{pmatrix} \quad . \quad (7.8)$$

The important property is that  $\mathcal{H}$  is symmetric under the bilinear map:

$$(\mathcal{H}\psi, \phi) = (\psi, \mathcal{H}\phi) \quad , \quad (7.9)$$

in the proof of which the surface terms generated in the integration by parts exactly cancel against those in (7.7). The relation (7.9) is the analog of self-adjointness, and leads to the usual proof that for two eigenvectors,

$$(\phi_m, \phi_n) = 0 \quad (7.10)$$

whenever  $\omega_m \neq \omega_n$ . Provided that  $(\phi_n, \phi_n) \neq 0$  [31], one can normalize these eigenfunctions in the usual way, i.e., by requiring (7.10) to be  $2\omega_m\delta_{mn}$  in general. We shall henceforth call this property orthonormality (and (7.10) alone as orthogonality), with the understanding that reference is made to the bilinear map (7.7) rather than the usual inner product. It also follows trivially that, provided this orthonormal system is complete (which is the case under fairly broad assumptions, see Section I), then the time evolution of a wavefunction  $\psi(t)$  is given by

$$\psi(t) = \sum_n a_n \phi_n e^{-i\omega_n t} \quad , \quad (7.11)$$

where the  $a_n$  are obtained by projecting the initial data:

$$a_n = \frac{(\phi_n, \psi(t=0))}{(\phi_n, \phi_n)} \quad . \quad (7.12)$$

The preservation of orthonormality under SUSY should therefore be sought in terms of the bilinear map (7.7).

### C. Normalization factor for QNMs under SUSY transformation

We first present a derivation of orthonormality that does not explicitly require the two-component formalism. With orthogonality in the sense of the bilinear map already guaranteed by (7.10), it remains only to compute the normalizing factor

$$\begin{aligned} (\tilde{\phi}_n, \tilde{\phi}_n) &= 2\omega_n \int_{-a}^a [(\partial_x + W)\phi_n]^2 dx \\ &\quad + i \left[ \tilde{\phi}_n(-a)^2 + \tilde{\phi}_n(a)^2 \right] \quad . \end{aligned} \quad (7.13)$$

Integrate by parts to convert  $(\partial_x \phi)^2$  to  $-(\partial_x^2 \phi)\phi$  plus a surface term, express the second derivative in terms of  $V - \omega_n^2$  by means of the eigenvalue equation, and write the potential as  $V = W^2 - W' + \Omega^2$ . Then, apart from a term proportional to  $\omega_n^2 - \Omega^2$ , the integrand becomes a total derivative  $\partial_x(W\phi^2)$ . Using  $W(\pm a)^2 = -\Omega^2$  and  $\partial_x \phi_n(\pm a) = \pm i\omega_n \phi_n(\pm a)$  then leads to

$$\frac{(\tilde{\phi}_n, \tilde{\phi}_n)}{(\phi_n, \phi_n)} = \omega_n^2 - \Omega^2 \quad . \quad (7.14)$$

Incidentally, the case of NMs on  $[-a, a]$  (which corresponds to nodal conditions at the ends of the interval) can be recovered by simply dropping all surface terms.

Since the ratio (7.14) takes the same form as for NMs, and is the eigenvalue of  $A^\dagger A$ , we can again write the normalized SUSY transformation for each eigenfunction as (7.3).

### D. SUSY transformation for two-component form

We have so far deliberately avoided the explicit two-component form of the SUSY transformation, since, for an eigenfunction  $\phi_n$ , one can always express  $\partial_t \phi_n$  in terms of  $\phi_n$ . However, in order to perform SUSY transformations on general outgoing wavefunctions (e.g., when given a time-dependent state, to find its SUSY partner at all times), the second component must be considered.

Since the SUSY transformation must commute with time-evolution and  $\phi^2 = \partial_t \phi^1$ , it is clear that both components must be transformed in the same way. Thus, the (unnormalized) SUSY transformation on the two-component vector is  $\mathcal{A} = \text{diag}(A, A)$ , which satisfies

$$(\mathcal{A}\psi, \phi) = (\psi, \mathcal{A}^\dagger \phi) \quad , \quad (7.15)$$

where  $\mathcal{A}^\dagger \equiv \text{diag} (A^\dagger, A^\dagger)$ . In deriving (7.15), one has to integrate by parts. In the integrand,  $\mathcal{A}$  turns into  $\mathcal{A}^\dagger$  because  $A$  contains a single derivative  $\partial_x$ , which reverses sign upon integration by parts. The surface terms are seen to work out by using, e.g.,  $\phi^2(a) = \partial_t \phi^1(a) = -\partial_x \phi^1(a)$ , and the known values of  $W_\pm$ .

Note that  $\mathcal{A}^\dagger \mathcal{A} = (H - \Omega^2)\mathbb{1}$ ,  $\mathcal{A}\mathcal{A}^\dagger = (\tilde{H} - \Omega^2)\mathbb{1}$ , i.e., these products do not relate to the two-component  $\mathcal{H}$ .

With (7.15), it is straightforward to show that the normalized SUSY transformation

$$\phi \mapsto \tilde{\phi} = \mathcal{A}(\mathcal{A}^\dagger \mathcal{A})^{-1/2} \phi \quad , \quad (7.16)$$

defined on the subspace orthogonal to  $\Phi$ , preserves the bilinear map, in a manner that exactly parallels (7.4). In the exceptional case of SUSYs that generate a doubled mode (see Section VIII), the subspace has to exclude *two* states on which  $H - \Omega^2$  vanishes.

These properties show that two apparently unrelated concepts, namely SUSY and the linear-space structure for QNMs in open systems (e.g., the replacement of inner products by the bilinear map), turn out to be consistent. This should not be surprising: the bilinear map (7.7) has an intrinsic geometric meaning for all outgoing states, not just QNMs [43].

## VIII. DOUBLED MODES

### A. Examples

In this section, we deal with the possibility that doubled modes may be produced by a SUSY transformation. Consider a system  $H$  with an NM  $\Phi(x)$  at  $\Omega = iK$  and a QNM  $\Psi(x)$  at  $-\Omega = -iK$ . Then both  $\tilde{\Phi}(x) = \Phi(x)^{-1}$  and  $\tilde{\Psi}(x) = A\Psi(x)$  are QNMs of  $\tilde{H}$  at  $-\Omega$ . Since there can only be *one* QNM at any frequency [31],  $\tilde{\Psi}(x) \propto \tilde{\Phi}(x)$ . The proportionality constant can be evaluated by

$$\begin{aligned} \frac{\tilde{\Psi}}{\tilde{\Phi}} &= \frac{\tilde{\Psi}(-a)}{\tilde{\Phi}(-a)} = 2i\Omega\Psi(-a)\Phi(-a) \\ &= \frac{\tilde{\Psi}(a)}{\tilde{\Phi}(a)} = -2i\Omega\Psi(a)\Phi(a) \quad . \end{aligned} \quad (8.1)$$

Incidentally, the agreement of these two expressions can also be seen without invoking SUSY. First one notes that  $\Phi$  and  $\Psi$ , being eigenfunctions of  $\mathcal{H}$  with distinct eigenvalues, are orthogonal. In the expression for the bilinear map, the integral term vanishes because the frequencies are opposite, leaving only the surface terms. Thus one finds  $0 = (\Psi, \Phi) = i[\Psi(-a)\Phi(-a) + \Psi(a)\Phi(a)]$ .

Secondly, consider a perturbation of  $H$  which splits the eigenvalues (originally both  $\Omega^2$ ) of  $\Phi$  and  $\Psi$ , in which case their counterparts  $\tilde{\Phi}(x) = \Phi(x)^{-1}$  and  $A\Psi(x)$  must also split into two states. Thus the one state represented by  $A\Psi \propto \Phi^{-1}$  must be the limit of two states that coalesce when the splitting is switched off, and in some sense to be made precise below must count as a *doubled* state.

This situation has already been encountered. Consider a PT potential with  $q = 1$ . Its modes include (see (5.5)) the NM  $\Phi$  at  $\omega_0^- = i/2$  and the QNM  $\Psi$  at  $\omega_1^- = -i/2$ , both with eigenvalue  $\Omega^2 = -1/4$ . Use  $\Phi$  to generate the SUSY partner with  $\tilde{q} = 0$ . Its QNMs  $\tilde{\Phi}$  at  $\tilde{\omega}_0^+$  and  $\tilde{\Psi} = A\Psi$  at  $\tilde{\omega}_0^-$  both occur at the same eigenvalue  $-1/4$ . (Other pairs also coalesce, but this particular pair is special in that one of the two states is the reciprocal of the generator.) This pair of states form a doubled mode, and we have verified that  $\tilde{\Psi} \propto \tilde{\Phi}$  (cf. (8.1)) indeed holds in this example.

As another example, consider a system  $H$  with a TTM<sub>L</sub>  $\Phi(x)$  at an eigenvalue  $\Omega^2$ . The partner system  $\tilde{H}$  must have a TTM<sub>R</sub>  $\tilde{\Phi}(x) = \Phi(x)^{-1}$  at the same eigenvalue. But if  $H$  should have a TTM<sub>R</sub>  $\Psi(x)$  also with the eigenvalue  $\Omega^2$ , then  $\tilde{\Psi}(x) = A\Psi(x)$  is a TTM<sub>R</sub> of  $\tilde{H}$  with the eigenvalue  $\Omega^2$  as well. As before, we expect these states to be proportional, and a calculation analogous to (8.1) shows that in this case

$$\begin{aligned} \frac{\tilde{\Psi}}{\tilde{\Phi}} &= -2i\Omega\Psi(-a)\Phi(-a) \\ &= -2i\Omega\Psi(a)\Phi(a) \quad . \end{aligned} \quad (8.2)$$

Again, this situation has been encountered. Consider the example in Figures 4 and 5. Since  $V$  is symmetric, it is guaranteed that for every TTM<sub>L</sub>  $\Phi(x)$  there will also be a TTM<sub>R</sub>  $\Psi(x) = \Phi(-x)$  at the same eigenvalue. With this substitution for  $\Psi$ , the condition (8.2) is readily verified for any symmetric potential.

## B. Double zeros of Wronskians

In this subsection we make precise what is meant by a doubled mode and in the next subsection we describe the SUSY transformation of such doubled modes. We shall do so for QNMs, the case of TTMs differing only by some signs. In the original system  $H$ , define outgoing-wave solutions of the wave equation  $f(\omega, x)$  and  $g(\omega, x)$ , satisfying the boundary conditions

$$\begin{aligned} f(\omega, x < -a) &= e^{-i\omega x} \quad , \\ g(\omega, x > a) &= e^{i\omega x} \quad , \end{aligned} \quad (8.3)$$

where  $' = \partial_x$ . (For TTMs,  $e^{i\omega x}$  on the r.h.s. also for  $f$ .) A QNM is a solution that satisfies *both* the left boundary condition (as on  $f$ ) *and* the right boundary condition (as on  $g$ ); in other words, at a QNM frequency,  $f$  and  $g$  are linearly dependent. Thus, the zeros of the Wronskian

$$\begin{aligned} J_q(\omega) &\equiv J(f, g; \omega) \\ &= f'(\omega, x)g(\omega, x) - f(\omega, x)g'(\omega, x) \end{aligned} \quad (8.4)$$

identify the QNMs. In fact, it is easy to show that the Wronskian is related to the transmission amplitude introduced earlier by:

$$J_q(\omega) = -\frac{2i\omega}{T_L(\omega)} \quad . \quad (8.5)$$

Now consider the analogous construction in the partner system  $\tilde{H}$ , obtained by using an NM  $\Phi$  of  $H$  as the generator. By our convention,  $\Phi$  is associated with a frequency  $\Omega$  on the positive imaginary axis, and  $W(\pm a) = -\Phi'(\pm a)/\Phi(\pm a) = \mp i\Omega$ . The SUSY transformation gives

$$\begin{aligned} \tilde{f}(\omega, x) &= (\partial_x + W)f(\omega, x) \\ \tilde{g}(\omega, x) &= (\partial_x + W)g(\omega, x) \quad , \end{aligned} \quad (8.6)$$

leading to the Wronskian

$$\begin{aligned} \tilde{J}_u(\omega) &= \{ [(\partial_x + W)f]' [(\partial_x + W)g] \\ &\quad - [(\partial_x + W)f] [(\partial_x + W)g]' \} \quad . \end{aligned} \quad (8.7)$$

This Wronskian is however unnormalized (as indicated by the subscript), since only  $C\tilde{f}$  and  $D\tilde{g}$  satisfy the normalization conventions at  $-a$  and  $+a$  respectively, where

$$C = -D = \frac{i}{\omega - \Omega} \quad . \quad (8.8)$$

These normalizing factors have no zeros or poles in the lower half-plane, which is the only region of interest for QNMs. Thus the normalized Wronskian in the partner system is

$$\begin{aligned} \tilde{J}_q(\omega) &= CD\tilde{J}_u(\omega) \\ &= (\omega - \Omega)^{-2}\tilde{J}_u(\omega) \quad . \end{aligned} \quad (8.9)$$

When (8.7) is written out, some terms cancel by using  $J'_q = 0$ , the second derivatives can be eliminated by the defining equation, and  $V$  is expressed in terms of  $W$  and  $\Omega^2$ . Some arithmetic then leads to

$$\tilde{J}_u(\omega) = (\omega^2 - \Omega^2) J_q(\omega) \quad , \quad (8.10)$$

$$\tilde{J}_q(\omega) = \frac{\omega + \Omega}{\omega - \Omega} J_q(\omega) \quad , \quad (8.11)$$

which is also readily derived from (4.3), since (8.5) shows that  $J_q$  transforms as  $T_L^{-1}$  under SUSY.

Eq. (8.11) neatly summarizes the properties of Type 1 SUSY transformations (cf. Table 1): the QNMs of  $\tilde{H}$  and  $H$  are the same except that the former has an extra QNM at  $\omega = -\Omega$ . In fact, this formula can be used in the upper half-plane as well, where the denominator in (8.11) indicates that the NM at  $\omega = \Omega$  is absent in  $\tilde{H}$ . Moreover,

(7.14) on the change in normalization under SUSY emerges as a simple consequence of (8.10), since the bilinear map is related to the Wronskian by [19]

$$(f(\omega_n), g(\omega_n)) = - \left[ \frac{\partial J_q(\omega)}{\partial \omega} \right]_{\omega_n} . \quad (8.12)$$

Now specialize to the case where  $H$  happens to have a QNM with eigenvalue  $\Omega^2$ ; this means  $J_q(\omega)$  has a zero at  $\omega = -\Omega$ . By (8.11),  $\tilde{J}_q$  has a *double* zero at this point. This then gives a precise definition of a doubled mode. It is clear that upon perturbation, such a double zero would generically split up into two simple zeros, i.e., two QNMs. The existence and properties of double (and higher) zeros in the Wronskian for QNMs relate to the issue of Jordan blocks (JBs), which has been extensively discussed recently [31], and to which we will return in Section VIII C.

With TTMs, the situation is entirely analogous, and the double zero refers to a Wronskian defined with boundary conditions appropriate to the type of TTMs under discussion. To be specific, consider a SUSY transformation of Type 3a, which deletes a  $\text{TTM}_L$  and produces a  $\text{TTM}_R$  (by convention we only consider TTMs in the lower half-plane). We define Wronskians using the boundary conditions appropriate to the latter, and in exactly the same way arrive at (8.11). To interpret the prefactor, we note that a  $\text{TTM}_R$  at  $\pm\Omega$  is the same as a  $\text{TTM}_L$  at  $\mp\Omega$ ; thus (8.11) in this case simply states that a  $\text{TTM}_L$  in the lower (i.e., a  $\text{TTM}_R$  in the upper) half-plane is deleted while a  $\text{TTM}_R$  in the lower (i.e. a  $\text{TTM}_L$  in the upper) half-plane is produced.

SUSY transformations of Wronskians are discussed in more detail and generality in Appendix B 5, in the context of potentials with tails.

The coalescence of two modes is similar to degeneracy (though one of the degrees of freedom is not an eigenstate). Usually, a degeneracy is associated with some symmetry. It is therefore interesting to note that in our example of doubled TTMs, the  $\text{TTM}_L$  and  $\text{TTM}_R$  are at the same frequency because of parity; under SUSY, they then turn into a doubled  $\text{TTM}_R$  in the partner system (even though the latter has no parity invariance).

### C. SUSY partner of a Jordan block

Returning to the situation described at the beginning of Section VIII A, let us examine in some detail the relationship between the doubled QNM at  $-\Omega$  of  $\tilde{H}$  and its SUSY pre-image.

To conform with the notation of Ref. [31], denote the state  $\Psi$  as  $\Psi_j$ , where  $j$  is a QNM index. Now, using the normalization convention (8.3), one has  $\Psi_j(x) = \Psi_j(-a)f(-\Omega, x)e^{i\Omega a}$ , implying  $\tilde{\Psi}_j(x) = \Psi_j(-a)\tilde{f}(-\Omega, x)e^{i\Omega a}$ . (Analogous formulas involving  $\Phi$  and  $\tilde{\Phi}$  follow by the proportionality (8.1) and will not be given.) Since  $\tilde{J}_u(\omega)$  has a double zero at  $\omega = -\Omega$ ,  $\tilde{f}(\omega, x)$  satisfies the outgoing condition at  $x = a$  not only at  $\omega = -\Omega$ , but also to first order away from the pole. This makes it plausible that, in the QNM expansion,  $\partial_\omega \tilde{f}(\omega, x)|_{-\Omega}$  takes the place of the “missing” eigenfunction when  $\tilde{\Phi}$  and  $\tilde{\Psi}_j$  coincide, which has been confirmed in detail [31]. One thus defines, for an arbitrary  $\alpha$ , a pair of functions  $\tilde{\Psi}_{j,n}$ , where the QNM index  $j$  labels the JB and  $n = 0, 1$  is an intra-block index:

$$\begin{aligned} \tilde{\Psi}_{j,0}(x) &= \tilde{\Psi}_j(x) \\ \tilde{\Psi}_{j,1}(x) &= \Psi(-a)\partial_\omega \tilde{f}(\omega, x)|_{-\Omega}e^{i\Omega a} + \alpha \tilde{\Psi}_j(x) \end{aligned} \quad (8.13)$$

where the second function satisfies

$$(\tilde{H} - \Omega^2)\tilde{\Psi}_{j,1} = -2\Omega\tilde{\Psi}_{j,0} . \quad (8.14)$$

Using this, one verifies that a time-dependent outgoing solution is  $\tilde{\Psi}_{j,1}(t) \equiv (\tilde{\Psi}_{j,1} - it\tilde{\Psi}_{j,0})e^{i\Omega t}$ , showing that the associated momentum reads  $\tilde{\Psi}_{j,1}^2 = -i(-\Omega\tilde{\Psi}_{j,1} + \tilde{\Psi}_{j,0})$ . One of the issues to be investigated is the behaviour of the term  $te^{i\Omega t}$  in  $\tilde{\Psi}_{j,1}$  under (inverse) SUSY, since such a time dependence cannot occur for  $H$ .

Next consider the bilinear map for these functions; because the second components are nontrivial, we explicitly use the two-component notation. For a double pole one always has  $(\tilde{\Psi}_{j,0}, \tilde{\Psi}_{j,0}) = 0$ , cf. (8.12). The undetermined constant  $\alpha$  in (8.13) can be used to also achieve  $(\tilde{\Psi}_{j,1}, \tilde{\Psi}_{j,1}) = 0$ ; this is useful in wavefunction expansions, but for our purposes  $\alpha$  can be left arbitrary.

The above merely applies standard JB theory to the double pole in the  $\tilde{H}$ -system. Turning now to SUSY transformations, one can trivially write

$$\begin{aligned} \tilde{\Psi}_{j,1}(x) &= \Psi(-a) \times \partial_\omega [(\partial_x + W)f(\omega, x)]|_{-\Omega}e^{i\Omega a} \\ &= (\partial_x + W)\Psi(-a)\partial_\omega f(\omega, x)|_{-\Omega}e^{i\Omega a} . \end{aligned} \quad (8.15)$$

Thus,  $\Psi(-a)\partial_\omega f(\omega, x)|_{-\Omega}e^{i\Omega a}$  is the SUSY pre-image of  $\tilde{\Psi}_{j,1}(x)$ . However, the former is *not* outgoing, since in the  $H$ -system the Wronskian only has a first-order zero at  $-\Omega$ , and consequently  $f(\omega, x)$  satisfies the outgoing-wave condition at  $x = +a$  only *at*  $-\Omega$ , and not to first order away from the eigenvalue.

The normalization of a JB is determined by one overall factor, which equals the bilinear map between the two basis states [31]. In the present case it is evaluated as

$$\begin{aligned} \frac{(\tilde{\Psi}_{j,1}, \tilde{\Psi}_{j,0})}{(\Psi_j, \Psi_j)} &= \left[ \frac{-\frac{1}{2}\partial_\omega^2 \tilde{J}_u(\omega)}{-\partial_\omega J_q(\omega)} \right]_{-\Omega} \\ &= -2\Omega \quad , \end{aligned} \quad (8.16)$$

where in the numerator of the first line we have used the result analogous to (8.12) for a double pole [31].

Let us finally consider the reverse transform generated by  $-A^\dagger$  [44]. It is easy to establish the following three properties. (a)  $A^\dagger \tilde{\Psi}_{j,0} \propto A^\dagger \tilde{\Phi} = 0$ . (b) Hence in  $A^\dagger \tilde{\Psi}_{j,1}(t)$ , the term  $\propto te^{i\Omega t}$  is annihilated. (c) The remaining term in  $A^\dagger \tilde{\Psi}_{j,1}$  is  $c\Psi_j$ . The last property is readily seen by observing that

$$\begin{aligned} (H - \Omega^2)(A^\dagger \tilde{\Psi}_{j,1}) &= A^\dagger(\tilde{H} - \Omega^2)\tilde{\Psi}_{j,1} \\ &= A^\dagger(-2\Omega\tilde{\Psi}_{j,0}) \\ &= 0 \quad , \end{aligned} \quad (8.17)$$

so that  $A^\dagger \tilde{\Psi}_{j,1}$  is an eigenfunction of  $H$  with eigenvalue  $\Omega^2$ . We have also noted that the SUSY pre-image of  $\tilde{\Psi}_{j,1}$  (i.e.,  $\partial_\omega f|_{-\Omega}$ ) is not an outgoing solution at all. These remarks completely resolve the puzzle related to the prefactor  $t$  in the time evolution in the  $\tilde{H}$ -system.

It is instructive to compare various calculations of the constant of proportionality  $c$  above. Defining  $\bar{\Psi}_j = A^\dagger \tilde{\Psi}_{j,1}$  (to avoid double stacked tildes), in the two-component formalism one can calculate

$$\begin{aligned} (\bar{\Psi}_j, \bar{\Psi}_j) &= (\tilde{\Psi}_{j,1}, \mathcal{A}\mathcal{A}^\dagger \tilde{\Psi}_{j,1}) \\ &= (\tilde{\Psi}_{j,1}, [\tilde{H} - \Omega^2]\mathbb{1}\tilde{\Psi}_{j,1}) \\ &= -2\Omega(\tilde{\Psi}_{j,1}, \tilde{\Psi}_{j,0}) \\ &= 4\Omega^2(\Psi_j, \Psi_j) \quad . \end{aligned} \quad (8.18)$$

One can get the same result from (8.10), for which one has to realize that in  $\bar{J}_u(\omega) = (\omega^2 - (-\Omega)^2)\tilde{J}_q(\omega)$ , one overall factor  $(\omega + \Omega)^2$  is associated with  $A^\dagger \tilde{f}(-\Omega, x) = 0$  and should be removed. Thus, one should evaluate

$$\begin{aligned} \frac{(\bar{\Psi}_j, \bar{\Psi}_j)}{(\tilde{\Psi}_{j,1}, \tilde{\Psi}_{j,0})} &= \left\{ \frac{-\partial_\omega[(\omega + \Omega)^{-2}\bar{J}_u(\omega)]}{-\frac{1}{2}\partial_\omega^2 \tilde{J}_q(\omega)} \right\}_{-\Omega} \\ &= -2\Omega \quad . \end{aligned} \quad (8.19)$$

For simple poles, one also knows the sign left undetermined by the bilinear map:  $\bar{\phi}_n \equiv A^\dagger \tilde{\phi}_n = A^\dagger A\phi_n = (\omega_n^2 - \Omega^2)\phi_n$ . For JBs, the corresponding calculation is

$$\begin{aligned} \bar{\Psi}_j &= A^\dagger A\Psi(-a)\partial_\omega f|_{-\Omega}e^{i\Omega a} \\ &= \Psi(-a)\partial_\omega[A^\dagger A f]_{-\Omega}e^{i\Omega a} \\ &= \Psi(-a)\partial_\omega[(H - \Omega^2)f]_{-\Omega}e^{i\Omega a} \\ &= \Psi(-a)\partial_\omega[(\omega^2 - \Omega^2)f]_{-\Omega}e^{i\Omega a} \\ &= -2\Omega\Psi_j \quad . \end{aligned} \quad (8.20)$$

Incidentally, if the eigenvalues  $\omega_j^2$  of the JB and  $\Omega^2$  of the SUSY auxiliary function are not equal, the transformation leaves the JB structure unmodified. An example of this situation has already been encountered in the PT potential with  $q = 1$ , for the QNMs with  $\omega_j^2 \neq -\frac{1}{4}$ . We omit the details, which are straightforward, as is the generalization to higher-order multiple poles.

## IX. DISCUSSION

In this paper, we have extended the usual discussion of SUSY as a relation between NMs of partner systems to include the QNMs as well; TTMs also come in naturally. By viewing all these together, a more complete picture emerges, most conveniently summarized by Table 1 or say (8.11). For example, in the usual discussion for NMs only, essentially isospectral systems are described as being identical except for one NM present in one system but absent in the other; now we see that (under Types 1 and 2 transformations) when an NM appears (disappears), a corresponding QNM disappears (appears).

QNMs differ from NMs in two regards. First, they have complex frequencies; nevertheless, even with twice as many constraints, matching the spectra is not any more difficult. Second, QNMs need not have an increasing number of nodes, and it is often possible to find several nodeless QNMs which generate distinct SUSY transformations — whereas the analogous operation for NMs would be restricted to the nodeless ground state.

The general formalism also provides a framework in which a number of well-known results find a convenient and unified expression. These include (Types 1 and 2) SUSY transformations relating different PT potentials of the same width, as well as (Type 4) SUSY transformations that take the free field into a nontrivial PT potential. The latter property provides one way of understanding the total-transmission property (at all positive energies) of a class of PT potentials. The transformation between the RW and Zerilli potentials can also be placed into this framework.

This wider perspective is gained only because attention is paid to the Klein–Gordon rather than the Schrödinger equation, since the concept of outgoing waves has no meaning in an equation that is first-order in time.

SUSY also preserves the orthonormality of QNMs in the sense of the bilinear map — supporting the view that both the extension of SUSY and the generalization of the inner product are sensible and useful.

Two further important properties are also preserved. (a) If  $V$  has a singularity say at  $x = \pm a$  (e.g., a step), then  $\tilde{V}$  will have the same type of singularity (but with the opposite sign). This is illustrated by many of the figures, and can be seen from (2.1) by noticing that the most singular part is  $W'$ . (b) If  $V$  has no tail, then for transformations of Types 1–3,  $\tilde{V}$  likewise has no tail. These two properties are precisely the necessary conditions for the QNMs (plus any possible NMs) to be complete [19]. Thus, SUSY maps a complete basis to a complete basis, if for Types 1 and 2  $\Phi \mapsto \tilde{\Phi}$  is included as well.

Finally, this work partially answers the question of QNM *inversion*: does one set of QNM (again plus NM) frequencies (roughly speaking carrying twice the information of one set of NM frequencies in closed systems) uniquely determine  $V$ ? In general the answer is negative, for there can be strictly isospectral potentials: see Types 3a and 3b in Table 1, and also Figure 5. However, if we consider a half-line problem  $x > 0$  (say the radial variable in a 3-d system), imposing a nodal condition at  $x = 0$  and the outgoing-wave condition at  $x > a$ , can one set of QNMs uniquely determine  $V$ ? SUSY transformations (1.5) do not directly rule out this scenario — for which there is some numerical evidence [24] — since these one-sided systems do not feature TTMs generating strictly isospectral partners. Moreover, by (2.1) and (2.3) the nodal condition maps a regular  $V$  to  $\tilde{V} \sim 2/x^2$  for  $x \rightarrow 0^+$  (generalizing the result that SUSY increases the angular momentum by one unit in the hydrogen atom). It would therefore be interesting to see if an enlarged class of transformations can address this question.

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## APPENDIX A: TRANSFORMATION BETWEEN THE WAVE AND KLEIN–GORDON EQUATIONS

In several previous papers [12], we have already commented on the relation between the KGE studied in the main text and the wave equation (WE) (A1) below. However, this relation exhibits some subtleties in the presence of bound states, or when the domain is the full line. Since the main text has shown both of these possibilities to be relevant to SUSY, in this appendix the transformation between the two equations will be investigated in detail.

Consider the WE

$$[\rho(z)\partial_t^2 - \partial_z^2]\psi(z, t) = 0 \quad . \quad (\text{A1})$$

In mechanics and optics,  $\rho$  is the mass density and the refractive index squared respectively, and thus is positive. Violation of  $\rho > 0$  would also make the time evolution singular and would take one outside the class of hyperbolic equations; we therefore exclude this. Then, one can define  $x$  by  $dx/dz = n(z) \equiv \sqrt{\rho(z)}$ ,  $\phi \equiv \sqrt{n}\psi$ , and

$$V = \frac{1}{2n^3} \partial_z^2 n - \frac{3}{4n^4} (\partial_z n)^2, \quad (\text{A2})$$

upon which  $\phi(x, t)$  is seen to satisfy the KGE (1.1). The transformation determines  $x$  up to an immaterial additive constant, which can for instance be fixed by letting  $z = 0$  map to  $x = 0$ . Clearly, if  $\rho(z) \rightarrow 1$  for  $|z| \rightarrow \infty$  then  $V(x) \rightarrow 0$  for  $|x| \rightarrow \infty$  (excluding pathologies).

The WE  $\mapsto$  KGE transform thus is well-behaved at infinity, but singularities in  $\rho$  require caution. While a kink in  $\rho$  leads to an admissible  $\delta$ -peak in  $V$ , already a step in  $\rho$  is too bad:  $\int V dx = \int V n dz = n'/2n^2 + \int (n')^2/4n^3 dz$ , with the second term diverging if  $n$  tends to a step through a sequence of smooth functions. The difficulty stems from the fact that both the WE and the KGE become distributionally undefined if  $\rho$  (or  $V$ ) is more singular than a  $\delta$ -function. The latter is the marginal case, in which the  $\delta$ -function multiplies a wave function which is not differentiable but still continuous.

Consider the two-component formalism. Since the WE  $\mapsto$  KGE transform leaves  $\partial_t$  invariant, one has  $\phi^2 = \dot{\phi}^1 = \sqrt{n}\psi^1 = n^{-3/2}\psi^2$ . For the bilinear map this leads to

$$\begin{aligned} & (\phi_1, \phi_2)_{\text{KGE}} \\ &= i \left[ \int_{\alpha}^{\beta} (\phi_1^1 \phi_2^2 + \phi_1^2 \phi_2^1) dx + \phi_1^1(\alpha) \phi_2^1(\alpha) + \phi_1^1(\beta) \phi_2^1(\beta) \right] \\ &= i \left[ \int_{-a}^a (\sqrt{n} \psi_1^1 n^{-3/2} \psi_2^2 + n^{-3/2} \psi_1^2 \sqrt{n} \psi_2^1) n dz \right. \\ &\quad \left. + \psi_1^1(-a) \psi_2^1(-a) + \psi_1^1(a) \psi_2^1(a) \right] \\ &= (\psi_1, \psi_2)_{\text{WE}}, \end{aligned} \quad (\text{A3})$$

where  $\alpha$  and  $\beta$  are the images of  $-a$  and  $a$  under the  $z \mapsto x$  map respectively, and where in the surface terms we have used the fact that  $n$  must be continuous for the transformation to exist. Thus, the WE  $\mapsto$  KGE map leaves the linear structure for open systems invariant.

When investigating the inverse transformation, the form (A2) is not immediately useful since there is no variable  $z$  to start with. Simple rewriting yields  $Hq = 0$ ,  $dz/dx = q(x)^{-2}$ ,  $\phi = q\psi$  with  $\rho = n^2 = q^4$ . The boundary condition to be used is  $\lim_{x \rightarrow \infty} q = 1$ . As long as  $q > 0$ , one sees that  $n > 0$  and hence  $\rho > 0$  as well. However, at a zero of  $q$ ,  $dz/dx$  suffers a non-integrable singularity and the transformation breaks down. For  $V > 0$  (in field theory, in the absence of broken symmetry  $V = m^2 > 0$ ) this never happens, while if  $V(x) < 0$  for some  $x$ , the transformation always exists locally but not necessarily globally.

More precisely, the transformation exists iff  $H$  has no bound state. (For the KGE, the main text shows that these lead to outgoing solutions which grow exponentially in time, evidently having no WE counterparts.) Namely, the breakdown of the transformation means that the bounded  $\omega = 0$  wave function has a node, implying that there is an integrable eigenstate with lower  $\omega^2$ .

For a KGE defined on the positive half-line (with a nodal condition on the fields at the origin), naturally the case in which  $q(x)$  has a single node at  $x = 0$  is marginal, and  $0 < x < \infty$  can just be mapped onto the entire  $z$ -axis. For problems defined on the full line, another subtlety shows up for  $x \rightarrow -\infty$ , where even in the absence of nodes in  $q$  the transformation is regular only if  $H$  has a mode at  $\omega = 0$  (cf. Appendix B6), i.e., if  $q$  tends to a constant for  $x \rightarrow -\infty$  as well. For all other  $V$  (which includes all nontrivial nonnegative potentials),  $q$  grows linearly for  $x \rightarrow -\infty$ , and the  $x$ -axis maps to a semi-infinite  $z$ -interval with a singularity in  $\rho$  at the lower boundary. Of course, by supplementing the differential equation for  $q$  with different boundary or initial conditions, in this latter case one can also map the  $x$ -axis onto the negative- $z$  semi-axis, or onto a bounded  $z$ -interval.

## APPENDIX B: POTENTIALS WITH TAILS

### 1. Preliminaries

Determining outgoing waves when the potential has a tail is delicate. Namely, whereas the boundary condition (1.10) is readily implemented at finite  $x = \pm a$  whenever these enclose the support of  $V$ , for potentials with tails the

condition can only be imposed asymptotically. However, if  $\text{Im } \omega < 0$  then  $\lim_{x \rightarrow \infty} \phi'(x)/\phi(x) = i\omega$  for *any* solution at eigenvalue  $\omega^2$  except the (unique up to a prefactor) small one, so that this criterion does not identify the outgoing wave.

If  $V$  vanishes at infinity faster than any exponential, one can impose the boundary condition at some  $x = L$  to find an approximate outgoing wave, which will converge to the exact one as  $L \rightarrow \infty$ . In practice, for any required accuracy one thus can simply truncate  $V$ , so that this case is still essentially the same as the one with no tail.

If however  $V(x) \sim e^{-\lambda x}$ , the above will only work for  $\text{Im } \omega > -\lambda/2$ , thus identifying only QNMs within a band below the real axis. By using the  $m$ th-order Born approximation for the asymptotic region, one can expand this band to  $i m \omega > -(m+1)\lambda/2$  [33].

For more heavily damped QNMs and for potentials which vanish sub-exponentially (i.e.,  $\lambda = 0$  in the above), sometimes (semi-)analytic solutions enable one to continue the outgoing wave from the upper into the lower-half  $\omega$ -plane, cf. Section [refonewave](#). Fortunately, this is the case for the PT, RW, and Zerilli potentials. In general, however, one needs sophisticated numerical techniques such as phase integrals, which involve integrating the KGE along carefully chosen paths in the  $x$ -plane cited.

## 2. One outgoing wave

For general potentials the calculation of outgoing waves thus can present difficulties; in fact already their definition needs some care. Therefore define solutions of the KGE  $f(\omega, x)$  and  $g(\omega, x)$  such that for  $\text{Im } \omega > 0$ ,  $f(\omega, x) \sim 1 \cdot e^{-i\omega x}$  for  $x \rightarrow -\infty$  and  $g(\omega, x) \sim 1 \cdot e^{i\omega x}$  for  $x \rightarrow \infty$ . This makes  $f$  and  $g$  analytic in this half-plane, and they can be analytically continued to the whole  $\omega$ -plane, with a cut on the negative imaginary axis. Functions  $\propto f$  and  $\propto g$  are said to be outgoing to the left and right, respectively, at frequency  $\omega$ ; a function is incoming at  $\omega$  iff it is outgoing at  $-\omega$ .

As a consequence, one will have  $g(-\omega^*, x) = g(\omega, x)^*$  and  $g(\omega, x \rightarrow \infty) \sim 1 \cdot e^{i\omega x}$  for generic points in the whole  $\omega$ -plane. It can happen that the latter relation does not hold at isolated points  $\omega'$  where  $g$  diverges; we will call such frequencies *special points* and write locally  $g(\omega) = (\omega - \omega')^{-n} \chi(\omega)$ , with  $\chi(\omega)$  finite near  $\omega = \omega'$  and  $\chi(\omega') \neq 0$ , and with an integer  $n \geq 1$ , to be called the *index* of the special point. Near the (possible) branch point  $\omega = 0$  the behaviour of  $g$  can be more complex, and it also can happen that  $\lim_{\omega \rightarrow 0} g(\omega) = 0$ .

From the Born series for  $g$  one sees that, if  $V(x \rightarrow \infty) \sim e^{-\lambda x}$ , frequencies  $\omega' = -ik\lambda/2$  ( $k = 1, 2, \dots$ ) can be special points: depending on higher-order contributions to  $V$ , the divergence in  $g$  could cancel. Examples of such “miraculous” cancellations include the PT potential, where the special points generically present in  $f(\omega, x)$  given by (5.2) at  $\omega = -in/b$  ( $n = 1, 2, \dots$ ) are absent if  $q = \frac{1}{2} + \ell$  ( $\ell = 0, 1, \dots$ ) and  $n \geq \ell + 1$ , as follows from standard properties of the hypergeometric function [45]; for  $\ell = 0$  this becomes the free field, which obviously has no special points at all. Without proof we mention that such a “miracle” also occurs at the algebraically special frequency (6.7) of the RW equation for any angular momentum. Special points need not lie on the imaginary axis, as is clear by considering  $V(x) \sim e^{-\lambda x} \cos(\mu x)$ ; however, only if  $\text{Re } \omega' = 0$  can  $\omega'^2$  be the eigenvalue of the SUSY generator, so this case needs to be studied most carefully. An example in which the special point index  $n > 1$  is provided by  $V(x) \sim x e^{-\lambda x}$  with  $n = 2$ , with obvious generalizations.

Physically, the divergence in  $g$  at a special point is irrelevant: it merely comes from the normalization imposed on  $g$ . Thus, this divergence cancels against the Wronskian in the Green’s function, the proper outgoing wave near  $\omega = \omega'$  in fact being  $\chi(\omega)$ . Since  $\chi(\omega) \sim (\omega - \omega')^n e^{i\omega x}$ , the large part of  $\chi$  vanishes at the special point. Because  $\chi(\omega)$  is a solution of the KGE with eigenvalue  $\omega^2$ , this can only happen if the outgoing wave  $\chi(\omega')$ , said to be *special outgoing*, at the same time is incoming as well:  $\chi(\omega') \propto g(-\omega')$ . While this might be counterintuitive, it merely means that at these frequencies the exponential tail scatters so strongly that the true outgoing wave is completely different from the one in free space. All this can be verified beyond the Born approximation for the exactly solvable cases  $V(x) = e^{-\lambda x}$  [15] and the PT potential of Section [V](#).

On the negative imaginary axis, the analytic continuation in general yields two different branches,  $g_l(\omega) = \lim_{\epsilon \downarrow 0} g(\omega - \epsilon)$  and  $g_r(\omega) = \lim_{\epsilon \downarrow 0} g(\omega + \epsilon) = g_l(\omega)^*$ .

The type of discontinuity this induces is indicated for each  $\omega' = -i|\omega'|$  by a second index  $m$  so that  $\Delta g(\omega) \equiv g_r(\omega) - g_l(\omega) \sim (\omega - \omega')^m$ . The case in which there is no cut at all can be thought of as  $m = \infty$ . Note that because of their normalization, the “large” parts of  $g_{l/r}$  cancel in  $\Delta g$ , which, also being a solution of the KGE, must thus be small if it does not vanish:  $\Delta g(\omega) \propto g(-\omega)$ , with a purely imaginary (frequency-dependent) constant of proportionality. At a special point,  $\chi_l$  and  $\chi_r$  must be proportional, their prefactors having opposite phases, and the cut index is then defined by  $\Delta \chi(\omega) \sim (\omega - \omega')^m$ .

The above treatment might seem to put undue emphasis on exceptions. However, these cases will actually arise by performing SUSY transforms on a generic potential, as will become clear in the following. The statements in this subsection have obvious counterparts for  $f$ .

### 3. SUSY transformation of one outgoing wave

We first consider how the outgoing wave at one end is transformed under SUSY. The crucial feature to be derived is: *even in the presence of tails, SUSY carries outgoing waves into outgoing waves unless it annihilates them*. The latter exception can only occur if the eigenvalues  $\omega^2$  of the outgoing wave and  $\Omega^2$  of the SUSY generator coincide, for  $Ag = 0$  iff  $g \propto \Phi$ . The proof is in fact simple. By (1.7), SUSY carries solutions of the wave equation into solutions for the partner system. For  $\text{Im } \omega > 0$ , the operator  $A$  of (1.6) carries exponentially decreasing waves into exponentially decreasing waves since  $W$  is bounded. Since the frequency independent  $A$  maps analytic functions to analytic functions, clearly it commutes with the analytic continuation to the rest of the  $\omega$ -plane, completing the proof.

Normalization of the transformed wave is straightforward, with the result that the outgoing wave in the partner system reads

$$\tilde{g}(\omega) = \frac{Ag(\omega)}{i\omega + W_+} \quad (\text{B1})$$

for generic  $\omega$ . In particular, this means that special points, branch cuts etc. are all conserved under SUSY if  $\omega^2 \neq \Omega^2$ . For  $\omega^2 = \Omega^2$ , one has to study the limiting behaviour of (B1) more carefully. Writing  $\Omega = \pm iK$  ( $K > 0$ ), one has the following cases.

(a1)  $W_+ = K$  and  $-iK$  is not a special point. In this case  $\Phi \propto g(iK)$ , so that the zero in the denominator of (B1) cancels, as it has to because  $\tilde{g}(\omega)$  is necessarily regular for  $\text{Im } \omega > 0$ . Hence,  $\tilde{g}(iK) = \lim_{\omega \rightarrow iK} \tilde{g}(\omega) = -iAg_1(iK)$ , with  $g_k(iK) \equiv k!^{-1} \partial_\omega^k g(\omega)|_{iK}$ , cf. the notation of (8.13). The expression for  $\tilde{g}(iK)$  is nonzero because  $g_1(iK) \not\propto g(iK)$ , as follows from  $(H + K^2)g_1(iK) = 2iKg(iK) \neq 0$ .

Let the original system have a discontinuity index  $m$  at frequency  $-iK$ . Frequency differentiation of the KGE yields  $(H + K^2)g_k(-iK) = -2iKg_{k-1}(-iK) + g_{k-2}(-iK)$  on both branches, so that  $(H + K^2)\Delta g_k(-iK)$  vanishes for  $k = m$ , but is nonzero for  $k = m + 1$ . Thus,  $\Delta g_m(-iK) \propto g(iK)$  (since  $\Delta g_m(-iK)$  is always exponentially small, cf. at the end of Section B2) and hence  $\Delta \tilde{g}_k(-iK) = 0$  for  $0 \leq k \leq m$  (for  $k < m$  this follows simply from  $\Delta g_k(-iK) = 0$ ). However,  $\Delta \tilde{g}_{m+1}(-iK) \neq 0$  so that  $\tilde{m} = m + 1$ . In particular,  $\tilde{m} \geq 1$  so that at  $\omega = -iK$ , the cut discontinuity vanishes in the partner.

It is trivial that  $\tilde{\Phi}$  is exponentially large, but is it actually outgoing? The question is answered by a calculation of the Wronskian

$$J\left(g^{-1}(iK), Ag(-iK)\right) = 0 \quad , \quad (\text{B2})$$

for which one has to substitute (1.6) for  $A$  with  $W = -g'(iK)/g(iK)$  and use the KGE for  $g''$ . Thus, up to proportionality  $\tilde{\Phi}$  equals the unique outgoing wave  $\tilde{g}(-iK) = Ag_{1/r}(-iK)/2K$ .

(a2)  $W_+ = K$  and  $-iK$  is a special point with index  $n = 1$ . The situation at  $\omega = iK$  is identical to case (a1). Near  $\omega = -iK$ ,  $g(\omega) = \chi(\omega)/(\omega + iK)$  with  $\chi(-iK) \propto g(iK)$  (cf. Section B2), so that  $A\chi(-iK) = 0$  and the special point is lifted in the partner system, with  $\tilde{g}(-iK) = A\chi_1(-iK)/2K \neq 0$ , where the r.h.s. in general will be different on the two branches. In more detail, a calculation as for case (a1) shows that now  $\tilde{m} = m$ .

The counterpart to (B2) is evaluated to be

$$\begin{aligned} J\left(g^{-1}(iK), A\chi_1(-iK)\right) &= -2iK\chi(-iK)/g(iK) \\ &\neq 0 \quad , \end{aligned} \quad (\text{B3})$$

so that in this case  $\tilde{\Phi}$  is mixed in the partner system [46].

(a3)  $W_+ = K$  and  $-iK$  is a special point with index  $n \geq 2$ . The situation at  $\omega = iK$  is identical to case (a1). At  $\omega = -iK$  one calculates  $\tilde{n} = n - 1$  with  $\tilde{\chi}(-iK) = A\chi_1(-iK)/2K$  and  $\tilde{m} = m$ . The function  $\tilde{\Phi}$  is exponentially increasing, hence certainly not (special) outgoing. All these properties are seen to be analogous to case (a2), except that now the frequency  $-iK$  remains special in the partner system.

(b1)  $W_+ = -K$ ,  $-iK$  is not a special point and  $\Phi \propto g_l(-iK) = g_r(-iK)$  is outgoing. Near  $\omega = iK$  the transformation is completely regular and  $\tilde{g}(iK) = -Ag(iK)/2K \propto \tilde{\Phi}$ , since  $\tilde{\Phi}$  is an exponentially decreasing solution with eigenvalue  $-K^2$ .

Near  $\omega = -iK$ ,  $\tilde{g}(\omega) = Ag(\omega)/(i\omega - K) \rightarrow -iAg_l(-iK) = \tilde{g}(-iK)$ . This generalizes the situation of (8.17), where  $-iK$  is taken to be a double QNM. Setting  $\Delta g(\omega) \sim (\omega + iK)^m$  ( $m \geq 1$ ) leads to  $\Delta \tilde{g}(\omega) = -i(\omega + iK)^{m-1} A \Delta g_m(-iK) + \mathcal{O}[(\omega + iK)^m]$  where  $A \Delta g_m(-iK)$  is readily checked to be nonzero, so that  $\tilde{m} = m - 1$ .

(b2)  $W_+ = -K$ ,  $-iK$  is not a special point, and  $\Phi$  is mixed. Cf. case (b1): an exponentially increasing  $\Phi$  will always be mixed if there is a discontinuity across the cut, for in that case  $g_{l/r}(-iK)$  are not proportional to a real function. The situation at  $\omega = iK$  is identical to case (b1).

Near  $\omega = -iK$ , the zero in the denominator of (B1) is not compensated, so  $-iK$  becomes a special point with index  $\tilde{n} = 1$  in the partner system and  $\tilde{\chi}(-iK) = -iAg(-iK)$ . Note how the large parts of  $\Phi$  and  $g(-iK)$  coincide, leading to the annihilation of this part in  $Ag(-iK)$  and leaving only an exponentially decreasing function. One verifies that  $\tilde{m} = m$  for the discontinuity index.

Note that, if  $V$  for instance has finite support,  $\tilde{V}$  has a tail  $\sim e^{-2Kx}$  as in (2.7), while  $-iK$  is its only special point. Thus, at  $i\omega/K = 2, 3, \dots$  one has a cancellation of the kind called “miraculous” in Appendix B 2, but which apparently is not exceptional in the context of SUSY.

(b3)  $W_+ = -K$  and  $-iK$  is a special point. The situation at  $\omega = iK$  is identical to case (b1). Near  $\omega = -iK$ , the pole in (B1) leads to  $\tilde{n} = n + 1$  for the special point index with  $\tilde{\chi}(-iK) = -iA\chi(-iK)$ , while  $\tilde{m} = m$ .

One sees that, for each  $i = 1, 2, 3$ , the partner of a system in the (ai) category falls into case (bi) and vice versa. Thus, for each  $i$ , (ai) and (bi) transforms are each other’s inverse, and inspection of the above cases confirms that their effects are exactly opposite.

Again, the situation for  $x \rightarrow -\infty$  is wholly analogous.

#### 4. Two outgoing waves

We now combine the left and right boundary conditions to study modes. For this purpose, we introduce the two Wronskians

$$J_q(\omega) \equiv f'(\omega)g(\omega) - f(\omega)g'(\omega) \quad , \quad (B4)$$

$$J_t(\omega) \equiv f'(-\omega)g(\omega) - f(-\omega)g'(\omega) \quad , \quad (B5)$$

which keep track of (quasi)normal modes and total-transmission modes respectively. That is, for ordinary points of  $f$  and  $g$  an  $M$ th-order zero in  $J_q(\omega=\omega_0)$  signifies an  $M$ th-order QNM if  $\text{Im } \omega_0 < 0$ , and an  $M$ th-order NM if  $\text{Im } \omega_0 \geq 0$ . In the latter case, this of course means that the only possible nonzero value is  $M = 1$ , in which case  $\text{Re } \omega_0 = 0$ . Similarly, an  $M$ th-order zero in  $J_t(\omega=\omega_0)$  signifies an  $M$ th-order TTM<sub>L</sub> at  $\omega_0$ , or equivalently an  $M$ th-order TTM<sub>R</sub> at  $-\omega_0$ .

While QNMs can coexist with NMs, or TTM<sub>L</sub>s with TTM<sub>R</sub>s, at the same eigenvalue  $\omega^2$  (cf. Section VIII), (Q)NMs and TTMs clearly exclude each other unless there are special points, at which the various notions coincide.

Of course, if  $\omega_0$  is a special point of index  $n$  for  $f$  and  $n'$  for  $g$ , then (Q)NMs are given by zeros of  $(\omega - \omega_0)^{n+n'} J_q(\omega)$  and TTM<sub>L</sub>s by those of  $(\omega - \omega_0)^{n'} (\omega + \omega_0)^n J_t(\omega)$ .

For example, let  $g(\omega)$  have a special point with index 1 at  $\omega = \omega_0$ . Then at the eigenvalue  $\omega_0^2$  one can have: (i) no mode, or (ii) an NM and a TTM<sub>L</sub> of order  $1 + k$  ( $k = 0, 1, \dots$ ) or (iii) a QNM of order  $1 + \ell$  and a TTM<sub>R</sub> of order  $1 + k$  ( $k, \ell = 0, 1, \dots$ ), with at most one of  $k, \ell$  nonzero. As a second example, if  $\omega_0$  is an  $n = 2$  special point for  $f$ , then there can at most be a first-order TTM<sub>R</sub> at that frequency, since now the notions of TTM<sub>R</sub> and NM coincide up to second order and higher-order NMs cannot occur.

If at least one of the functions  $f$  and  $g$  has a cut on the negative imaginary axis, then  $J_q$  will have such a cut as well (similar to the cut in the Green’s function). On the other hand, clearly if there is a branch cut in  $f$  ( $g$ ) then  $J_t$  will have a cut on the positive (negative) imaginary axis. It follows from the symmetries satisfied by  $f$  and  $g$  individually, that if  $\phi_l$  is a zero-mode at frequency  $\omega_0$  on the left branch of  $J_q$ , then  $\phi_r = \phi_l^*$  is a zero-mode (of the same order  $M$ ) on the right branch. As a consequence, if  $f_l(\omega_0) = f_r(\omega_0)$  (for instance if  $f$  has no branch cut at all) then  $g(\omega_0)$  should be proportional to a real function; hence, if  $\omega_0$  is an ordinary point for  $g$  then  $\Delta g$  vanishes there ( $m_R \geq 1$ ). For TTMs the situation is simpler, since only one cut at a time is involved: if, say, there is a TTM<sub>L</sub> at  $\omega_0$ , then  $\max(m_R, n_R) \geq 1$ . Generalizations to higher-order modes can be derived, but are not given here.

## 5. SUSY transformation of two outgoing waves

With the six possibilities of Section B 3 for both  $x \rightarrow -\infty$  and  $x \rightarrow \infty$ , and with the rich variety of possible mode structures in the original system outlined in Section B 4, a classification of the possible SUSY transformations may seem daunting even before considering the subtleties of  $\Omega = 0$ . However, most of the information is contained in the transformation properties of the Wronskians (B4) and (B5). These are readily derived by substituting (B1) and its counterpart for  $\tilde{f}$  into (B4) for  $\tilde{J}_q$  and (B5) for  $\tilde{J}_t$ , using (1.6) for  $A$  and the KGE for  $f''$  and  $g''$ . One obtains

$$\tilde{J}_q(\omega) = \frac{(\omega + iK)(\omega - iK)}{(\omega + iW_-)(\omega - iW_+)} J_q(\omega) \quad , \quad (\text{B6})$$

$$\tilde{J}_t(\omega) = \frac{(\omega + iK)(\omega - iK)}{(-\omega + iW_-)(\omega - iW_+)} J_t(\omega) \quad . \quad (\text{B7})$$

Combining these with the transformation properties of the individual functions  $f$  and  $g$  derived in Section B 3, one finds that also the mode structure is entirely conserved if  $\omega^2 \neq \Omega^2$ , leaving only the case  $\omega^2 = \Omega^2$  to be studied.

For instance, if  $W_{\pm} = \pm K$  (an NM), then (B6) shows that in  $\tilde{J}_q$  a zero at  $iK$  gets cancelled. This is a much more meaningful statement than just noting that  $A\Phi = 0$  for the NM SUSY generator  $\Phi$ , which in itself says nothing about the situation in the partner system. At the same time, an extra zero at  $-iK$  appears. However, this latter zero must be interpreted carefully. If  $-iK$  is an ordinary point for both  $f$  and  $g$ , it will remain so in the partner and the zero signifies that a QNM at  $-iK$  is created (or its order increased). If  $-iK$  is special for either  $f$  or  $g$ , then the extra zero merely reduces  $n$  by one according to case (a2) or (a3) in Section B 3, and the order (possibly zero) of the QNM at  $-iK$  is *not* altered, consistent with  $\tilde{\Phi}$  *not* being outgoing on the ‘special’ side. Finally, if both  $f$  and  $g$  are special at  $-iK$  then the decrease of the associated indices by itself would reduce the pole in  $J_q$  by two orders, so that in this case (B6) shows that the NM-generated SUSY transform *reduces* the QNM index at  $-iK$  by one. In fact this should not be surprising, for if  $-iK$  is two-sided special then at least to first order the QNM agrees with the NM, which gets cancelled in the partner; an example is the inverse of the transform considered in Section V E.

Continuing with the example, one sees that  $J_t$  merely changes sign. Again, depending on the properties of  $f$  and  $g$ , this can mean no change in the mode situation (if  $-iK$  is ordinary), destruction of a TTM<sub>L</sub> (if  $-iK$  is special for  $g$ ), destruction of a TTM<sub>R</sub> (if  $-iK$  is special for  $f$ ), or both of the latter.

Generalizing from this example, one sees that the material of Section B 3 together with (B6) and (B7) leads to the following statement: *if  $-iK$  is an ordinary point for both the original and the partner system (which in particular implies that  $\Phi$  is purely incoming or outgoing on both sides), the only effect of SUSY on the mode structure consists of the destruction of the mode  $\Phi$  and the creation of the mode  $\tilde{\Phi}$ , in both cases possibly by changing their order by one.* Hence, in this case the situation remains as in Table 1 even in the presence of arbitrary potential tails.

It may seem that (B6) holds the possibility of creating a second-order NM by starting from a system with an NM at  $iK$ , and performing a SUSY transform using a generator for which  $W_{\pm} = \mp K$ . However, this can never occur because at the eigenvalue  $-K^2$ , any real solution except possibly the NM itself must have a node and hence cannot be used as a SUSY generator. For a proof which is also valid at  $K = 0$ , assume that  $\phi$  is the NM with  $\phi(x \rightarrow \infty) > 0$  for definiteness, while  $\tilde{\Phi} \not\propto \phi$  is a positive-definite solution at the same eigenvalue. As  $x \rightarrow \infty$ ,  $\tilde{\Phi}/\phi \rightarrow \infty$  so  $(\tilde{\Phi}/\phi)' = J(\tilde{\Phi}, \phi)/\phi^2 > 0$ . Since the Wronskian is position independent,  $(\tilde{\Phi}/\phi)' > 0$  also for  $x \rightarrow -\infty$ , which is only possible if  $\phi < 0$  there, implying that the function has at least one node. At the rightmost node,  $J(\tilde{\Phi}, \phi) = -\phi'\tilde{\Phi} < 0$ , and the contradiction with the position independence of  $J$  means that the function  $\tilde{\Phi}$  of the assumption does not exist.

## 6. SUSY at $\Omega = 0$

Throughout this appendix and the main text, it has been clear that there is an interplay between the outgoing waves and possible modes at  $iK$  and  $-iK$ . Hence, special attention should be paid to the case when these two frequencies coincide, i.e.,  $\Omega = 0$ . For the KGE at  $\omega = 0$ , it is still true that for  $x \rightarrow \infty$ , there is one unique (up to a prefactor) small solution. This same solution is also the outgoing wave  $\chi(\omega=0)$ , as follows by analytically continuing the small solution  $g(\omega)$  from the upper half  $\omega$ -plane to the origin (the same argument can *not* be used to conclude that  $\chi(\omega=0)$  should be asymptotically large by performing the continuation from the lower half plane, since at any frequency large functions are not unique, and the limiting member of a sequence of them could well be asymptotically small). The issue of special points does not arise at zero frequency, and neither does that of discontinuity indices.

For  $\Omega = 0$ , the six cases of Section B3 reduce to two: for  $x \rightarrow \infty$ ,  $\Phi$  can be either (a) outgoing, i.e. small, or (b) large. However, the properties of the transformation depend sensitively on the potential tail. Here we study  $V(x) \sim \kappa/x^\alpha$ , with  $\alpha \geq 2$  [47]; for the practically important case  $\alpha = 2$  we take  $\kappa = \nu(\nu+1) \geq -1/4$ , with  $\nu \geq -1/2$  without loss of generality. Tails which decay faster than algebraically (including no tail at all) are easy to handle, and can be thought of as  $\alpha = \infty$ .

For  $\alpha > 2$ , all terms in the Born approximation exist down to  $\omega = 0$ . To first order [15],  $g(0, x) = 1 + \kappa x^{2-\alpha}/(\alpha-1)(\alpha-2)$ , so in case (a) one has  $W(x) = \kappa x^{1-\alpha}/(\alpha-1)$ , i.e., SUSY reverses the sign of the potential tail. In the partner system,  $\tilde{g}(0) = \lim_{\omega \rightarrow 0} Ag(\omega)/i\omega = -i\partial_\omega Ag(\omega)|_0 \propto \tilde{\Phi}$ ; only if  $\alpha > 3$  can one write  $\tilde{g}(0) = -iAg_1(0) \propto \tilde{\Phi}$  for this, since  $g(\omega)$  itself is not differentiable at  $\omega = 0$  for  $2 < \alpha \leq 3$ . The proportionality  $Ag_1(-iK) \propto \tilde{\Phi}$  has not been encountered before in Section B3; Eq. (B3) shows why. In case (b),  $\Phi \sim x + \mathcal{O}(x^{3-\alpha}) + \mathcal{O}(x^0)$ , so  $W(x) \sim -1/x$ ,  $\tilde{V}(x) \sim 2/x^2$ , and  $\tilde{g}(\omega) = \chi(\omega)/\omega$  with  $\tilde{\Phi} \propto \chi(0) = -iAg(0) \sim i/x$ . Thus, for these potentials at  $\Omega = 0$ , *any* choice of  $\Phi$  leads to a small  $\tilde{\Phi}$ , in contrast to the situation at finite  $\Omega$ ; of course, the partner potential depends on this choice.

If  $\alpha = 2$ , one has the exact solution  $g(\omega, x) = e^{i(\nu+1)\pi/2} \sqrt{\pi\omega x/2} H_{\nu+1/2}^{(1)}(\omega x)$  [15] with  $H^{(1)}$  a Hankel function, i.e.,  $g(\omega, x) \sim [\Gamma(\nu+1/2)/\sqrt{\pi}](2i/\omega x)^\nu$  ( $\sim (i-1)\ln\omega\sqrt{\omega x/\pi}$  for  $\nu = -1/2$ ) near  $\omega = 0$ . In case (a), the generator thus is  $\Phi = x^{-\nu}$  and the transformation can be summarized by  $\tilde{\nu} = \nu - 1$  ( $-\nu$ ) for  $\nu \geq 1/2$  ( $-1/2 \leq \nu \leq 1/2$ ). In case (b), one has  $\Phi \sim x^{\nu+1}$  and  $\tilde{\nu} = \nu + 1$  [48]. Thus, one sees that for  $-1/2 \leq \nu \leq 1/2$ , always  $\tilde{\Phi} \propto \tilde{\chi}(0)$  (indeed, a large  $\tilde{\Phi}$  would give  $\tilde{\nu} \mapsto \tilde{\nu} + 1$  in the inverse transform, which for any  $\tilde{\nu}$  is incompatible with  $\nu < 1/2$ ) as for  $\alpha > 2$ . Such potentials, which at  $\Omega = 0$  behave essentially the same as one with finite support, have *weak tails*. On the other hand, for *strong tails* ( $\alpha = 2, \nu > 1/2$ )  $\tilde{\Phi}$  is large (small) if  $\Phi$  is small (large), as is the case for  $\Omega \neq 0$ .

In line with usage for other frequencies,  $\omega = 0$  is said to be a mode frequency if  $\chi(\omega=0) \propto \phi(\omega=0)$ , where  $\phi$  is the counterpart of  $\chi$  for waves outgoing to the left. Like any NM, such a mode is necessarily of first order, and there is no distinction between different types of modes. Apart from possible overall prefactors  $\phi(\omega)/f(\omega)$  and  $\chi(\omega)/g(\omega)$  due to the tails, a mode is associated with a *second-order* zero in  $J_{q/t}$  because the eigenvalue in the KGE is  $\omega^2$ .

If the original system has a mode at  $\omega = 0$ , the proof at the end of Section B5 implies that (if it is nodeless) the mode itself is the only candidate for a SUSY generator, i.e., case (a) applies on both sides. If  $V$  has two weak tails,  $\tilde{\Phi}$  will again be a mode; combination with the above statements on the transformation of one potential tail shows that the class of weak-tail potentials with a mode at  $\omega = 0$  is closed under SUSY (since transforms at  $\Omega \neq 0$  obviously cannot generate strong tails), as exemplified in Section VE by some of the potentials SUSY-equivalent to the free field. If  $V$  has a weak tail on one side,  $\tilde{\Phi}$  is outgoing on that side only so that there is no mode at  $\omega = 0$ . If  $V$  has two strong tails,  $\tilde{\Phi}$  is large on both sides. Moreover, according to the above in that case one has  $\tilde{\nu}_{L/R} = \nu_{L/R} - 1$  for the tail amplitudes, so that  $\tilde{f}(\omega)\tilde{g}(\omega)/f(\omega)g(\omega) \sim \omega^2$ ; since (B6) and (B7) show that up to a sign  $J_{q/t}$  are invariant under SUSY, the compensating factor  $\omega^{-2}$  must come from the disappearance of the mode.

If the original system has no mode at  $\omega = 0$  then the situation is simpler, even though several types of SUSY may be possible. If  $\tilde{\Phi}$  is outgoing on both sides (which in particular will always be the case if  $V$  has two weak tails) then it is obviously a mode itself. It is not iff  $\Phi$  is small on only one side and if  $V$  has a strong tail on that side. Then  $\tilde{\Phi}$  is small on only the other side, i.e., if  $\tilde{\Phi}$  is not a mode then  $\tilde{V}$  has no mode at all at  $\omega = 0$ .

To conclude, let us mention that modes, special points etc. at  $\omega^2 \neq \Omega^2 = 0$  are not affected by SUSY, as is the case for other generator eigenvalues.

## APPENDIX C: NODES IN QNM WAVEFUNCTIONS

Nodeless eigenstates play a special role in SUSY: they are candidates for the SUSY generator. It is therefore useful to highlight the differences between NMs and QNMs in this regard, especially to contrast with the well-known property that there can be *only one* nodeless NM.

First of all, we show that for QNMs not lying on the imaginary axis (i.e.,  $\text{Re}\omega \neq 0$ ), there can be at most one node or antinode. This is not surprising: since the eigenvalue is complex, the wavefunction has a changing phase, and it would be “unlikely” that the real part and the imaginary part (or their derivatives) would vanish together. To prove this, we take the Schrödinger point of view, so that the eigenvalue is  $\lambda = \omega^2$  with a nonzero imaginary part. Now consider a time-dependent QNM and suppose that it has nodes or antinodes at two points  $x_1, x_2$ . At these two points, the current

$$J = i[\phi^*(\partial_x \phi) - (\partial_x \phi^*)\phi] \quad (C1)$$

vanishes. Then, applying the conservation law in integral form to the interval  $[x_1, x_2]$ , we find that the total probability

in this interval is constant in time. Yet the wavefunction is either growing or decaying, since  $\text{Im } \lambda \neq 0$ , which is therefore a contradiction.

From the perspective of SUSY it is unfortunate that the above proof excludes the crucial imaginary axis. However, on that axis the statement remains valid for repulsive potentials, or more generally for potentials which are so weakly attractive that  $V - \omega^2$  is positive definite. Namely, let  $\phi$  be a solution with two (anti)nodes. By taking the real or imaginary part if necessary,  $\phi$  can be arranged to be real. Now between two nodes  $\phi$  would have an extremum, i.e.  $\phi''\phi < 0$  which is incompatible with the KGE. Similarly, an antinode can only be a global maximum or minimum, precluding the presence of any other nodes or antinodes. Note that for repulsive  $V$ , the KGE can be mapped to the wave equation (Appendix A; the mapping conserves nodes but not necessarily antinodes), for which a proof using the current  $J$  (cf. (C1)) does not exclude the imaginary axis.

Thus, QNMs (at least those off the imaginary axis for attractive  $V$ ) can have at most one node or antinode. For symmetric potentials, in the even sector  $x = 0$  is already an antinode, so there can be no nodes anywhere.

For zero modes, i.e., QNMs with  $\text{Re } \omega = 0$ , nodes are more “likely”: the eigenvalue is real and (if there is no branch cut) the wavefunction has a constant phase, say purely real; it therefore requires only *one* condition, rather than two, to have a node. Nevertheless, in contrast to the conservative case, the proof that there is at most one nodeless eigenstate can be bypassed.

The interlacing nodal structure of NM eigenfunctions follows from well-established Sturm–Liouville theory. For the present purpose, we do not need the full apparatus. Consider, for simplicity, a finite interval  $[-a, a]$  and suppose there are two distinct nodeless eigenfunctions  $\phi_1, \phi_2$ , both of which can be chosen to be real. Then they can both be chosen to be positive, which violates the orthogonality condition for NMs

$$\int_{-a}^a \phi_1(x)\phi_2(x) dx = 0 \quad . \quad (\text{C2})$$

We can attempt to transplant the argument to QNMs. For zero modes, again the wavefunctions have constant phases and can be chosen to be real, and if they are nodeless, positive definite. However, the analog of (C2) for two eigenfunctions with eigenvalues  $\omega_i = -i\gamma_i$  is

$$\begin{aligned} & -(\gamma_1 + \gamma_2) \int_{-a}^a \phi_1(x)\phi_2(x) dx \\ & + [\phi_1(-a)\phi_2(-a) + \phi_1(a)\phi_2(a)] = 0 \quad . \end{aligned} \quad (\text{C3})$$

Note in particular the signs of the two terms. With  $\gamma_i > 0$ , this condition does *not* preclude both eigenfunctions from being positive definite.

Thus, we have three different situations. (a) For NMs, there can be *only one* nodeless state. (b) For QNMs with  $\text{Re } \omega = 0$ , there *could be* more than one state with no node. (c) For QNMs with  $\text{Re } \omega \neq 0$ , each eigenfunction can have at most one node or antinode, and for symmetric potentials, *every* even eigenfunction is nodeless.

Case (b) in particular opens up the possibility of multiple SUSY transformations, as exemplified in Sections III B and V C.

## APPENDIX D: SELF-REPLICATING POTENTIALS

In Section V C we saw that the PT potential is self-replicating under SUSY, in the sense that the partner is the same potential with a different strength:

$$\tilde{V}(x) = \alpha V(x) \quad . \quad (\text{D1})$$

The condition (D1) is a special case of shape-invariant potentials described by a family  $U(a, x)$  [49], with

$$\begin{aligned} V(x) &= U(a, x) \\ \tilde{V}(x) &= U(\tilde{a}, x) + c(a) \quad . \end{aligned} \quad (\text{D2})$$

Here we focus on the more restrictive sense (D1) and show that the PT potential is uniquely determined by this condition. From (D1) and (2.1), and with  $\Omega = -iK$  being purely imaginary, we find

$$W'(x) = \beta [W(x)^2 - K^2] \quad , \quad \beta \equiv \frac{\alpha-1}{\alpha+1} \quad . \quad (\text{D3})$$

All nonsingular nonconstant solutions of (D3) differ merely by  $x$ -translation. The unique antisymmetric one reads

$$\begin{aligned} W(x) &= -K \tanh K\beta x \quad , \\ \Phi(x) &= (\cosh K\beta x)^{1/\beta} \quad . \end{aligned} \tag{D4}$$

Putting this into (2.1), we then find

$$\begin{aligned} V(x) &= (\beta-1)K^2 \operatorname{sech}^2 K\beta x \quad , \\ \tilde{V}(x) &= (-\beta-1)K^2 \operatorname{sech}^2 K\beta x \quad . \end{aligned} \tag{D5}$$

That is,  $V$  and  $\tilde{V}$  are of PT form.

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- [3] See, e.g., F. Cooper, A. Khare and U. Sukhatme, Phys. Rep. **251**, 267 (1995); G. Junker, *Supersymmetric Methods in Quantum and Statistical Physics* (Springer, Berlin, 1996) and references therein.
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- [7] In other papers, we have also considered NMs which are obtained as the limit of QNMs when the escape rate of the waves is tuned to zero, e.g., by considering potentials with a barrier of indefinitely increasing height at  $x \approx \pm a$ . These NMs are defined on finite intervals, and have positive eigenvalues  $\omega^2$ ; thus they are represented on the real  $\omega$  axis. In this paper we do *not* consider the limit where the rate of escape goes to zero, and therefore there is no danger of confusing NMs in this sense with the NMs on the positive imaginary axis in the  $\omega$ -plane.
- [8] If  $\operatorname{Im} \omega > 0$ , the eigenfunction is square-integrable, so the Hermiticity of  $H$  implies that  $\omega^2 \in \mathbf{R}$ .
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- [13] In calculating the Green's function  $G(t)$  by inverse Fourier transform of  $\tilde{G}(\omega)$  [12], one has to integrate along a line in the complex  $\omega$ -plane that lies above all the singularities, to ensure that  $G(t < 0) = 0$ ; for  $t > 0$  this then captures all the singularities, including the NMs.
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- [24] W. S. Lee, Thesis, The Chinese University of Hong Kong (1998).
- [25] Suppose one has a signal  $\phi(t) = \sum_n a_n \exp(-i\omega_n t)$  in terms of a set of (possibly complex) eigenfrequencies  $\omega_n$ . The coefficients  $a_n$  depend on the initial conditions, whereas the frequencies  $\omega_n$  are determined by the system (i.e., the potential  $V$ ). Thus, going through the frequencies is the natural way of filtering the information to retain only those properties intrinsic to the system.
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- [30] The nomenclature follows the NM case. It may seem paradoxical that the case where the spectra are imperfectly matched is said to be good, while the case of perfect matching is said to be broken; the reason stems from boson-fermion symmetry in field theory.
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- [43] A. Maassen van den Brink and K. Young, “Factor-space construction of outgoing waves”, Preprint (1999).
- [44] If the double pole is split along the imaginary axis, there are two distinct SUSY transforms generated by  $\tilde{\Phi}$  and  $\tilde{\Psi}_j$  respectively, which become identical when the poles merge. Thus, in the spirit of the JB approach to the field expansion, one might try to find the “missing” SUSY map generated by  $\tilde{\Psi}_{j,1}$ . However, no meaningful transformation emerges. This difference with the case of the field expansion can probably be understood by realizing that, when the poles split perpendicular to the imaginary axis upon changing the sign of the splitting perturbation, the SUSY transforms cease to exist altogether.
- [45] It is remarkable that the values of  $q$  for which these “miracles” occur are the same as those for which the PT potential has total transmission (Section V E), for the former property is completely determined by the asymptotics of the potential on one side, whereas the latter is global.
- [46] With the replacement  $g(\Omega) \mapsto \chi(\Omega)$ , the calculation leading to (B2) remains valid in cases a2 and a3. However, it does not convey information since on the l.h.s.  $A\chi(\Omega) = 0$ .
- [47] For  $\alpha < 2$ ,  $V(x \rightarrow \infty)$  does not behave as a small tail anymore. That  $\alpha = 2$  is the marginal case is caused by considering  $\omega = 0$  [15]. For finite  $\omega$ , the marginal case would be  $\alpha = 1$  (cf. the asymptotics of Coulomb wave functions), as can readily be seen from the WKB approximation.

- [48] At  $\nu = -\frac{1}{2}$ , the large generator is  $\Phi \sim \sqrt{x} \ln x$ , and there are significant sub-leading corrections in the  $\tilde{\nu} = \frac{1}{2}$  partner potential so that its small solution  $\tilde{\chi}(\omega=0)$  is  $\tilde{\Phi} = \Phi^{-1}$  instead of  $x^{-1/2}$ .
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| Type | $\Phi$                           | $\tilde{\Phi}$ | NM | QNM | TTM <sub>L</sub> | TTM <sub>R</sub> |
|------|----------------------------------|----------------|----|-----|------------------|------------------|
| 1    | DD                               | II             | -1 | +1  | 0                | 0                |
| 2    | II                               | DD             | +1 | -1  | 0                | 0                |
| 3a   | DI                               | ID             | 0  | 0   | -1               | +1               |
| 3b   | ID                               | DI             | 0  | 0   | +1               | -1               |
| 4    | more complicated, see Appendix B |                |    |     |                  |                  |

Table 1: The boundary conditions for the deleted state  $\Phi$  and the new state  $\tilde{\Phi}$ , and the changes in the number of NMs, QNMs, TTM<sub>L</sub>s and TTM<sub>R</sub>s in the SUSY transformation.

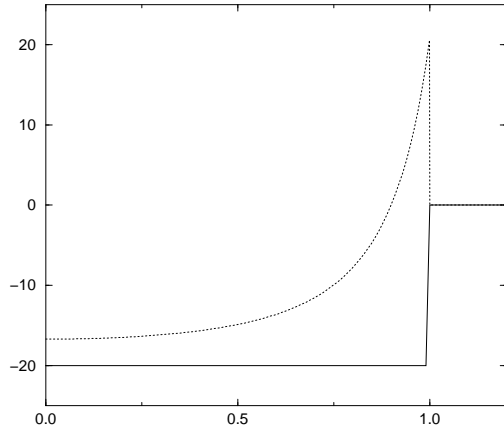


Fig. 1(a): A square-well potential  $V$  (solid line) defined by (3.1) with  $V_0 = -20$  and  $a = 1$ , and its SUSY partner potential  $\tilde{V}$  (broken line).

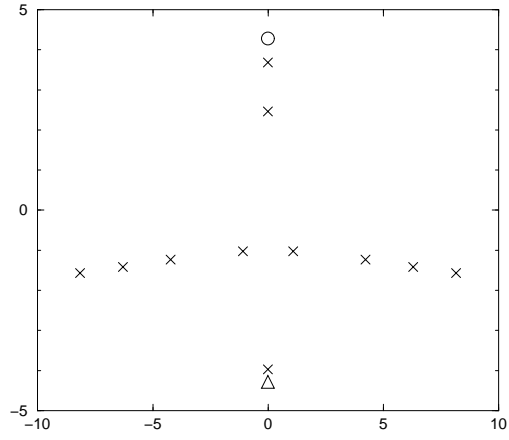


Fig. 1(b): The complex  $\omega$ -plane showing the NMs and QNMs common to both potentials (crosses); the mode present only in  $V$  (circle), which corresponds to the generator  $\Phi$ ; and the mode present only in  $\tilde{V}$  (triangle), which corresponds to  $\tilde{\Phi} = A\Phi$ .

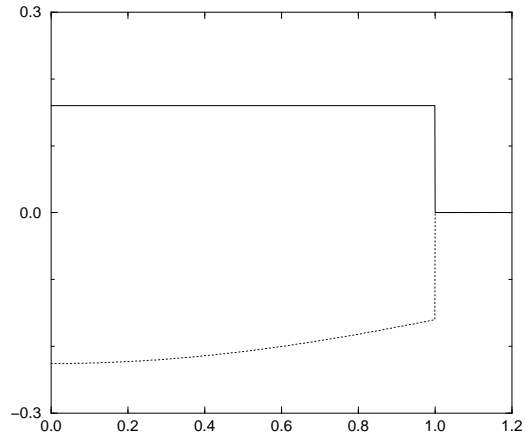


Fig. 2(a): A square-barrier potential  $V$  (solid line) defined by (3.1) with  $V_0 = 0.16$  and  $a = 1$ , and its SUSY partner potential  $\tilde{V}$  (broken line). The SUSY transformation is constructed by using the state at  $\omega_1 = -0.181i$  (circle in Fig. 2(b)) as the generator.

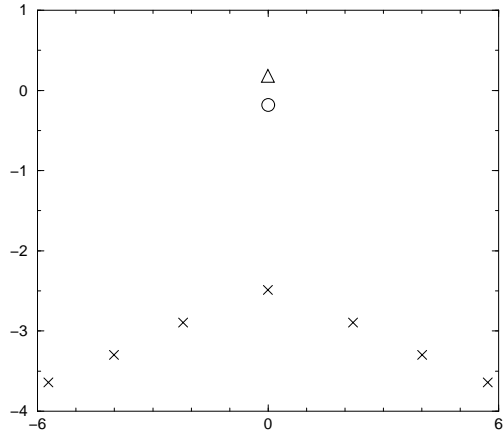


Fig. 2(b): The complex  $\omega$ -plane showing the QNMs common to both potentials (crosses); the mode present only in  $V$  (circle), which corresponds to the generator  $\Phi$ ; and the mode present only in  $\tilde{V}$  (triangle), which corresponds to  $\tilde{\Phi} = A\Phi$ .

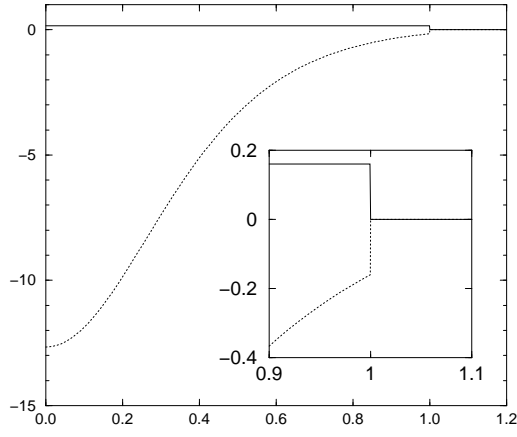


Fig. 3(a): A square-barrier potential  $V$  (solid line) defined by (3.1) with  $V_0 = 0.16$  and  $a = 1$ , and its SUSY partner potential  $\tilde{V}$  (broken line). The SUSY transformation is constructed by using the state at  $\omega_2 = -2.500i$  (circle in Fig. 3(b)) as the generator. Inset shows one portion enlarged.

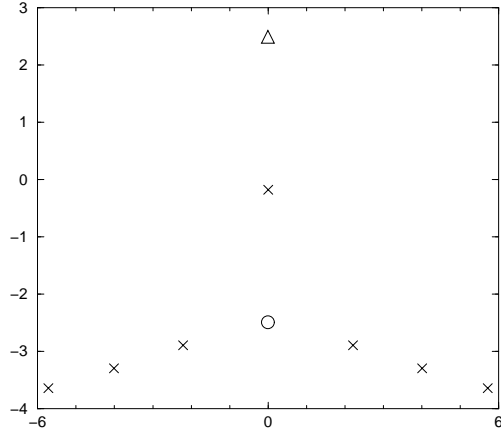


Fig. 3(b): The complex  $\omega$ -plane showing the QNMs common to both potentials (crosses); the mode present only in  $V$  (circle), which corresponds to the generator  $\Phi$ ; and the mode present only in  $\tilde{V}$  (triangle), which corresponds to  $\tilde{\Phi} = A\Phi$ .

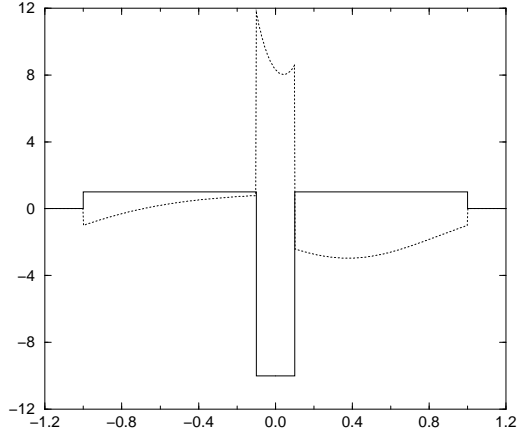


Fig. 4(a): A multi-step potential  $V$  (solid line) defined by (3.6) with  $V_0 = 1.0$ ,  $V_1 = -10.0$ ,  $a = 1.0$  and  $b = 0.1$ , and its SUSY partner potential  $\tilde{V}$  (broken line).

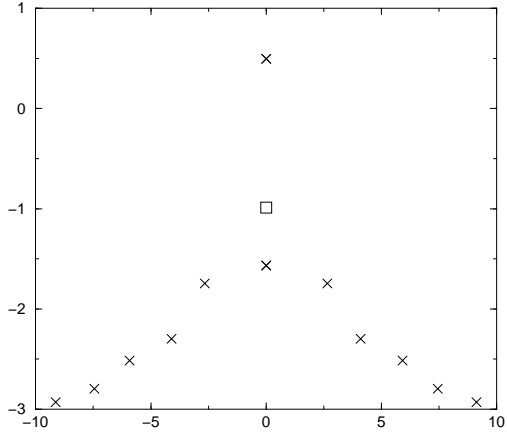


Fig. 4(b): The complex  $\omega$ -plane showing the NM and QNMs common to both potentials (crosses). The square indicates a  $\text{TTM}_L$  and a  $\text{TTM}_R$  in  $V$ , and a doubled  $\text{TTM}_R$  in  $\tilde{V}$ .

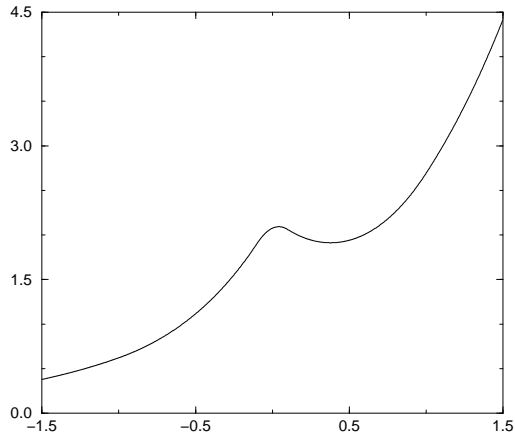


Fig. 5(a): The  $\text{TTM}_L$  of  $V$  used as the generator  $\Phi$  in the SUSY

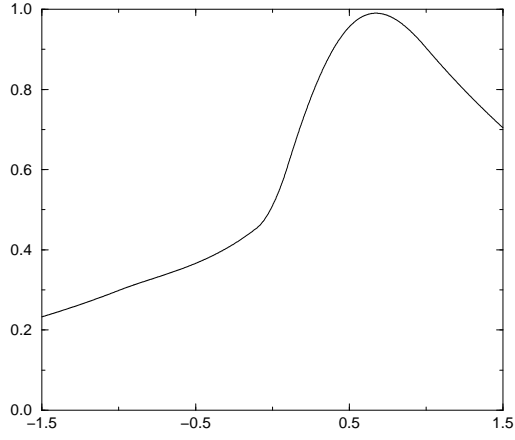


Fig. 5(b): The NM  $\tilde{\phi}_1$  in  $\tilde{V}$ , at  $\omega_1 = 0.498i$ .

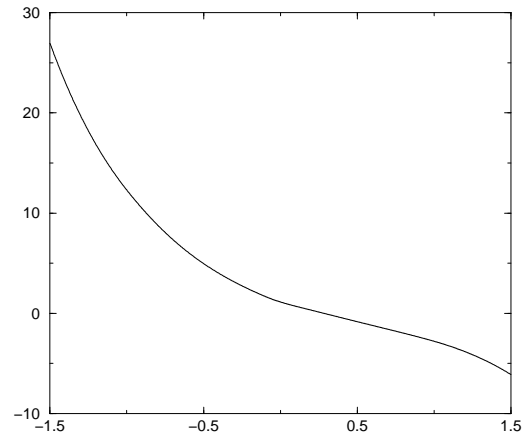


Fig. 5(c): A QNM  $\tilde{\phi}_2$  in  $\tilde{V}$ , at  $\omega_2 = -1.570i$ .

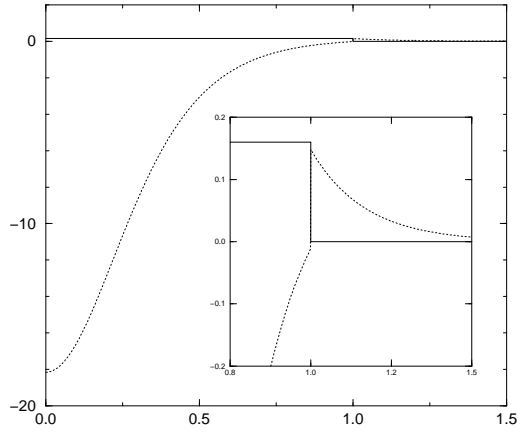


Fig. 6 (a): A square-barrier potential  $V$  (solid line) defined by (3.1) with  $V_0 = 0.16$  and  $a = 1$ , and its SUSY partner potential  $\tilde{V}$  (broken line) generated by a Type 4 transformation, using  $K = 3$ ,  $c = 1.0$  and  $d = -0.829$ . Inset shows one portion enlarged.

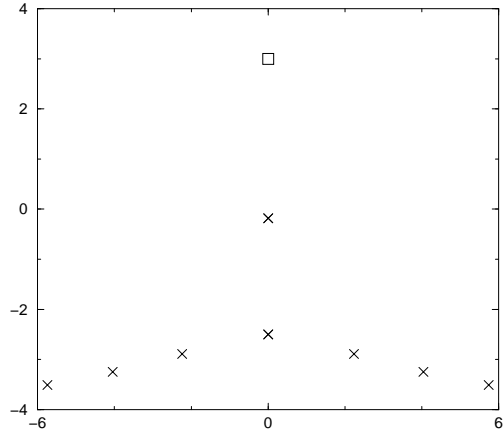


Fig. 6(b): The complex  $\omega$ -plane showing the QNMs common to both potentials (crosses); the mixed mode  $\Phi$  with boundary condition MM in  $V$ , used as the generator (square).

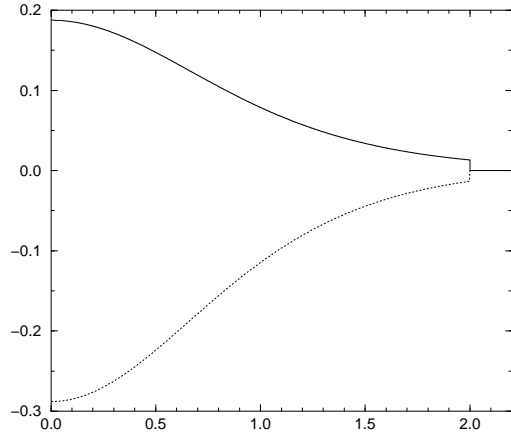


Fig. 7(a): The truncated PT potential (5.11) with  $\mathcal{V} = 3/16$ ,  $b = 1$ , and  $a = 2$ , and its SUSY partner potential  $\tilde{V}$  (broken line) generated by a Type 2 transformation, using the state at  $\Omega = -0.224i$ .

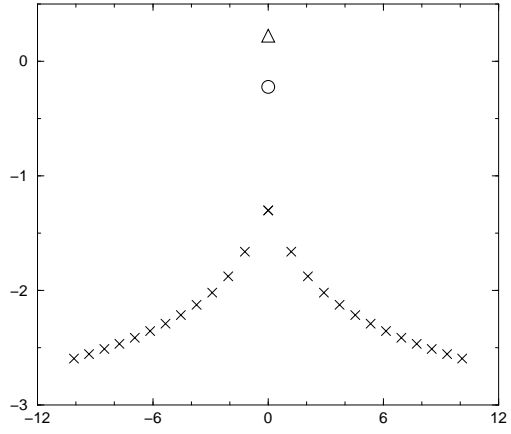


Fig. 7(b): The complex  $\omega$ -plane showing the QNMs common to both potentials (crosses); the mode present only in  $V$  (circle), which corresponds to the generator  $\Phi$ ; and the mode present only in  $\tilde{V}$  (triangle), which corresponds to  $\tilde{\Phi} = A\Phi$ .

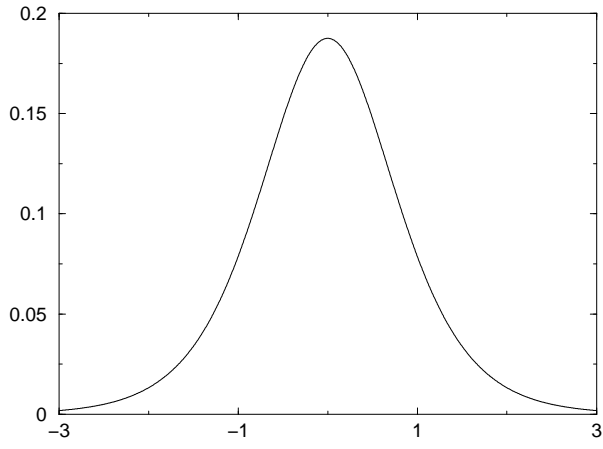


Fig. 8(a): The PT potential  $V(x)$  with  $\mathcal{V} = 3/16$ ,  $b = 1$ .

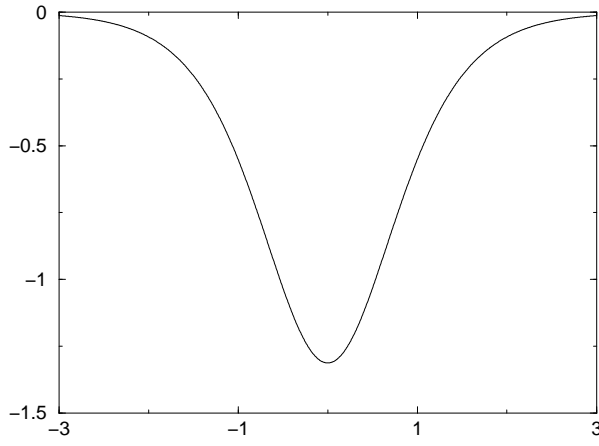


Fig. 8(b): Its SUSY partner  $\tilde{V}_0^+(x)$ . It is also a PT potential, with  $\tilde{\mathcal{V}} = -21/16$ .

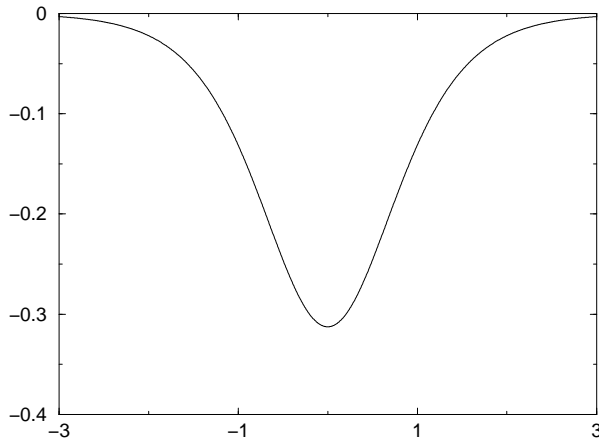


Fig. 8(c): Its SUSY partner  $\tilde{V}_0^-(x)$ . It is also a PT potential, with  $\tilde{\mathcal{V}} = -5/16$ .

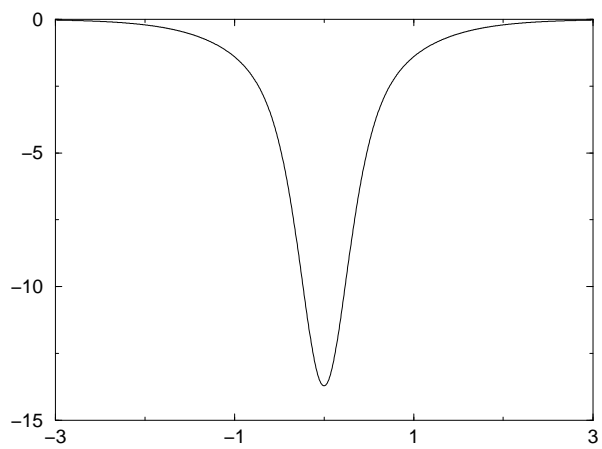


Fig. 8(d): Its SUSY partner  $\tilde{V}_2^+(x)$ .

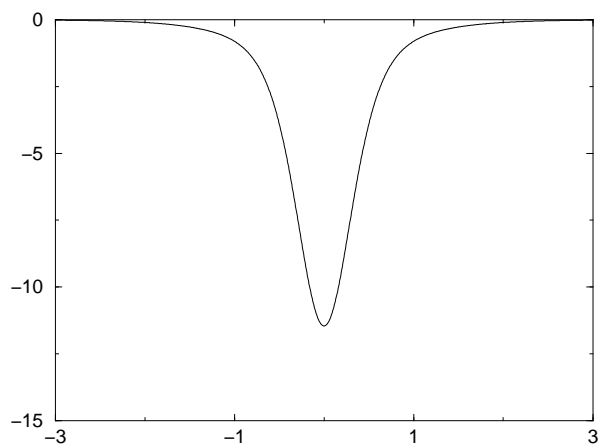


Fig. 8(e): Its SUSY partner  $\tilde{V}_2^-(x)$ .

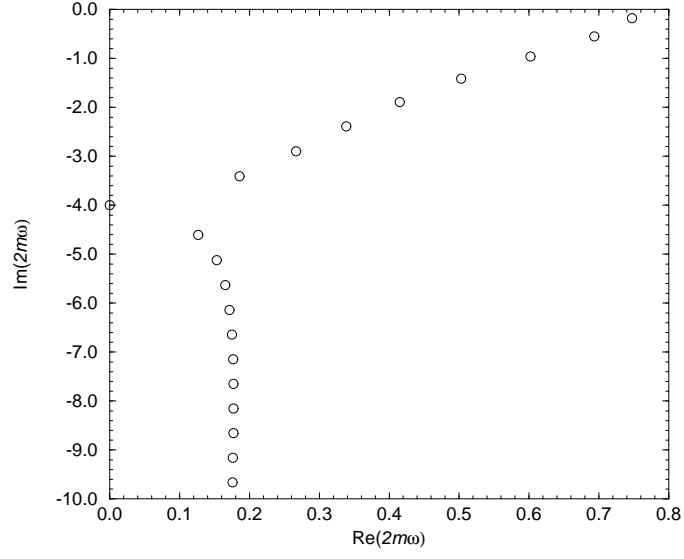


Fig. 9: The distribution of QNMs of a Schwarzschild black hole for  $l = s = 2$ . The mode on the imaginary axis  $2m\omega = -4i$  is not a QNM, but the special mode.